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Stephen Devlin
University of San Francisco, smdevlin@usfca.edu

T Treloar

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Cooperation in an evolutionary prisoner's dilemma on networks with degree-degree correlations

Stephen Devlin[1] and Thomas Treloar[2]
[1]*Mathematics Department, University of San Francisco, 2130 Fulton Street, San Francisco, California 94117, USA*
[2]*Mathematics Department, Hillsdale College, 33 East College Street, Hillsdale, Michigan 49242, USA*
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We study the effects of degree-degree correlations on the success of cooperation in an evolutionary prisoner's dilemma played on a random network. When degree-degree correlations are not present, the standardized variance of the network's degree distribution has been shown to be an accurate analytical measure of network heterogeneity that can be used to predict the success of cooperation. In this paper, we use a local-mechanism interpretation of standardized variance to give a generalization to graphs with degree-degree correlations. Two distinct mechanisms are shown to influence cooperation levels on these types of networks. The first is an intrinsic measurement of base-line heterogeneity coming from the network's degree distribution. The second is the increase in heterogeneity coming from the degree-degree correlations present in the network. A strong linear relationship is found between these two parameters and the average cooperation level in an evolutionary prisoner's dilemma on a network.

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I. INTRODUCTION

Agent-based game theoretical methods have become widely used tools in biology, social sciences, physics, and mathematics [1]. Such models are particularly appropriate in studying the dynamics of conflict and cooperation between individuals, making them especially relevant in all areas of biology, where an agent's fitness often depends on others with whom he interacts [2]. Recently, evolutionary models that deviate from traditional well-mixed population assumptions have become particularly prominent because of their ability to exhibit realistic behavior that is unsustainable in the well-mixed setting. The paradigm here is the prisoner's dilemma (PD), which is widely identified in many real-world situations [1–6], but where traditional game theoretical predictions often fail to accord with empirical behavior.

In the simplest form of the two-player PD, agents independently choose cooperation (C) or defection (D). Payoffs are associated with the four possible game states according to the symmetric matrix (where payoffs go to the row player)

$$\begin{array}{c|c|c} & C & D \\ \hline C & R & S \\ \hline D & T & P \end{array}. \tag{1}$$

Payoffs satisfy $T > R > P \geq S$, from which it follows that a rational self-interested agent has no motivation to play C: the payoffs for the D strategy strictly dominate those for the C strategy regardless of the coplayer's choice. The result is a Nash equilibrium where both players defect, the dilemma arising from the inefficiency of this equilibrium: both players would fare better with mutual cooperation [4].

Following the common practice begun in [7], payoffs are normalized by taking $R=1$ and $P=S=0$, so that the game depends only on the temptation to defect $T=b>1$. In the evolutionary version of the PD, interactions are repeated and agents update strategies based on their relative success. In the well-studied case of a fully mixed population, with strategy updating determined by the replicator dynamics, cooperation is quickly eradicated from the population [4]. Consequently, the dilemma persists and an explanation of how and why cooperation emerges in nature remains a fundamental cross-disciplinary problem. It is worth noting here that the repeated PD further requires that payoffs satisfy $T+S<2R$. Without this additional condition, full cooperation is no longer Pareto optimal as players can collectively obtain higher payoffs by alternating their individual strategies between C and D. With normalizations, the added condition is $1<b<2$.

Dropping the assumption of a well-mixed population, considerable attention has been focused on the discrete replicator dynamics of an evolutionary PD played on a network [7–25]. Vertices represent agents and edges represent contact information. Strategy updating depends on the success of an individual relative only to the success of that individual's neighborhood of contacts as specified by the network.

In this framework, the evolutionary dynamics are strikingly different. Nowak and May [7] famously showed that cooperation was sustainable on a lattice for certain game parameter values, paving the way for wide ranging investigations of the particular role played by the network topology on cooperation levels [7–15,22,23]. Recently, the sizable impact of scale-free networks on cooperation phenomena has been widely reported, and it has been shown that these networks are particularly hospitable to cooperators. In fact, cooperation can become the dominant strategy on a scale-free network even when the temptation to defect is very high [9,11]. In light of these findings, network heterogeneity has emerged as a key factor in the potential success of cooperators. Heterogeneity is widely understood to mean that the network contains considerable diversity in the numbers of agents' contacts, resulting in a degree distribution that is significantly "spread out" and includes large vertices or "hubs." Recent work has shown that mitigating the role played by these large vertices, through either payoff normalization [16] or participation costs [15], can dramatically reduce the success of cooperators, solidifying the notion of heterogeneity as a necessary ingredient in network cooperation. Even notions of heterogeneity that do not pertain to the static network itself have been shown effective in promoting cooperation in the PD. Recent studies include the roles played by social diversity [17,18], coevolution in the form of the ability of successful agents to reshape contact neighborhoods [19], and different teaching capacities among distinguished agents in the network [20]. For a survey of the research in the field, see [8].

In [21], the authors introduced a refined measure of network heterogeneity called standardized variance, denoted ν_{st}. An advantage of ν_{st} is that it gives a quantification of network heterogeneity that allows for comparisons of networks with different degree distributions and even different average degrees. Moreover, standardized variance has a convenient description in terms of the generating functions associated with the network's degree distribution. Through generating functions, ν_{st} interprets heterogeneity as a relationship between the number of connections of an average agent in the network and the number of connections of an average neighbor in the network. This quantification of heterogeneity gives additional insight into the mechanisms by which the network topology promotes cooperation. Focusing on the relative size difference between an average neighbor and an average agent uncovers a linear relationship between the average cooperation level in the population and a functional form of ν_{st} that is closely related to the network's epidemic threshold [26,27]. Once heterogeneity is quantified, it can be used to accurately predict cooperation levels on the networks in question.

The results in [21], however, hold on networks with specified degree distributions, but which are otherwise entirely random. In particular, the probability that a vertex of degree k has a neighbor of degree j is independent of k. It has been shown that many real-world networks fail to exhibit such independence [28]. In particular, correlations between the degrees of vertices at either end of an edge in the network are present in the widely studied cases of networks with power-law degree distributions generated by growth and preferential attachment [29–31].

In this paper, we consider an analog of standardized variance for networks with degree-degree correlations, including those generated by growth and preferential attachment. Building on the methods in [21], we develop a generalized notion of heterogeneity called correlated standardized variance, denoted ν_c, and apply it to the study of cooperation phenomena. If two networks have the same degree distribution, they might appear as equally heterogeneous. However, ν_c allows one to quantify the additional heterogeneity present in a correlated network coming from degree-degree correlations. That such correlations can have considerable impact on cooperation has been seen in [9] and is further evident in what follows. Moreover, it is shown that cooperation depends on correlated standardized variance in a way analogous to the results in [21]. The methods used facilitate comparisons across networks with varied degree distributions, heterogeneity, and average degrees. Finally, we also isolate the contributions of both ν_{st} and ν_c to the success of cooperation on a network, helping to quantify and to clarify the relationship between these network parameters, and in particular, helping to isolate the specific role played by degree-degree correlations in the evolutionary dynamics on the network.

II. GENERATING FUNCTIONS AND NETWORK HETEROGENEITY

A network $\mathcal{N}$ is an undirected graph of vertices connected by edges, in which neither loops nor multiple edges are allowed. Let X be the random variable that takes values in the set of possible degrees of vertices in the network. Following [32], let p_k be the probability that a random vertex in the network has degree k; i.e., p_k is the probability that X takes the value k. The probability generating function for the distribution of X is then given by

$$G(x) = \sum_{k>0} p_k x^k.$$

It follows that $G(1)=1$ and that $G'(1)=\langle k\rangle$ is the average degree of the vertices in the network.

While $G(x)$ captures the degree distribution of the network, all other specific contact information is ignored. Consequently, $G(x)$ can be thought of as representing a network chosen uniformly at random from the collection of all networks sharing the specified degree distribution.

Next, consider the distribution of degrees of vertices reached by choosing a random edge in the network. Let Y denote the associated random variable. A random edge is k times more likely to lead to a vertex of degree k than a vertex of degree of 1. The probability generating function of degrees of vertices reached along randomly chosen edges is therefore given by

$$T(x) = \frac{\sum_{k>0} k p_k x^k}{\sum_{k>0} k p_k} = \frac{1}{G'(1)} \sum_{k>0} k p_k x^k = \frac{x G'(x)}{G'(1)}. \qquad (2)$$

Finally, the notion of a randomly chosen neighbor in the network is defined as follows. First, a vertex is selected at random from the network followed by a randomly chosen edge emanating from the vertex. The degree distribution of vertices reached in this manner is the distribution of a *randomly chosen neighbor*. If the random variables X and Y are independent, then the distribution of randomly chosen neighbors is the same as the distribution of Y and so has a probability generating function $T(x)$. In this case the average degree of a random neighbor is given by $T'(1)$.

Following [21], the variance of the degree distribution is a natural first measure of network heterogeneity,

$$\mathrm{var}[X] = \langle k^2\rangle - \langle k\rangle^2,$$

where $\langle k\rangle$ denotes the expected value of X and $\langle k^2\rangle$ denotes the expected value of X^2. Since

$$\langle k\rangle = \sum_k k p_k = G'(1),$$

$$\langle k^2\rangle = \sum_k k^2 p_k = G'(1)T'(1),$$

it follows that

$$\mathrm{var}[X] = G'(1)T'(1) - G'(1)^2 = G'(1)[T'(1) - G'(1)].$$

Using the interpretations of $T'(1)$ and $G'(1)$ given above, the variance is the difference between the sizes of a randomly chosen neighbor and a randomly chosen vertex in the network, multiplied by the average network degree. In order to obtain a unitless measure of network heterogeneity, the variance is normalized by $G'(1)^2$ to give the uncorrelated standardized variance

$$\nu_{st} = \frac{T'(1) - G'(1)}{G'(1)}. \quad (3)$$

In the context of a particular network, therefore, ν_{st} is the difference between the average numbers of contacts of a randomly chosen neighbor and of contacts of a randomly chosen vertex, relative to the average number of contacts of a randomly chosen vertex. In [21], ν_{st} and its variants were used to study uncorrelated network heterogeneity and its effects on cooperation phenomena in an evolutionary PD.

Next, the assumption that X and Y are independent random variables is dropped. Consequently, $T(x)$ no longer needs to be the probability generating function for the distribution of degrees of randomly chosen neighbors in the network. Indeed, if the degree of a vertex reached along an edge emanating from a degree k vertex depends on k, we say that the network exhibits degree-degree correlations. However, Eq. (3) suggests a generalization of ν_{st} in the presence of such correlations provided that the actual average degree of a randomly chosen neighbor can be computed.

Consider, therefore, a network exhibiting degree-degree correlations. The probability that an edge leads from a degree k vertex to a degree j vertex is now determined by the conditional probability distribution $\{q_{kj}\}$. The network's nearest-neighbor function [23,26,33] is given by

$$k_{nn}(k) = \sum_j j q_{kj}. \quad (4)$$

The probability of choosing a vertex of degree k followed by an edge leading to a vertex of degree j is

$$p_k q_{kj}. \quad (5)$$

Write E_{RN} for the average degree of a randomly chosen neighbor in the (correlated) network. Summing over the network degree by degree, and using the probabilities in Eq. (5), gives

$$E_{RN} = \sum_{k,j} j p_k q_{kj} = \sum_k p_k k_{nn}(k). \quad (6)$$

Therefore, the average degree of a neighbor is the weighted average of the nearest-neighbor function k_{nn}, with the value $k_{nn}(k)$ weighted by p_k. In the absence of degree-degree correlations, $p_k q_{kj} = \frac{jp_j}{\langle k\rangle}$, and it follows that

$$E_{RN} = k_{nn}(k) = T'(1).$$

A correlated extension of standardized variance is now possible. Define ν_c to be the difference between the average degrees of a randomly chosen neighbor and randomly chosen vertex in the network, normalized by average network degree. It follows that

$$\nu_c = \frac{E_{RN} - \langle k\rangle}{\langle k\rangle}. \quad (7)$$

On an uncorrelated network, such as the one constructed from a degree distribution using the configuration model [34], Eq. (7) reduces to Eq. (3), and correlated standardized variance agrees with uncorrelated standardized variance: $\nu_c = \nu_{st}$.

It is worth noting that E_{RN}, k_{nn}, and ν_c can be understood in the context of the generating function formalism. With $\{p_k\}$ and $\{q_{jk}\}$ as above, one can consider a generating function in two variables as follows:

$$F(x,y) = \sum_{k,j>0} p_k q_{jk} x^k y^j.$$

Clearly $F(x,1)=G(x)$, the probability generating function for the degree distribution as given above. On the other hand, the probability generating function for the degrees of neighbors of a randomly chosen vertex is given by $F(1,y)$. If the degree random variable X, and the neighbor random variable Y, are independent, then $F(1,y)=T(y)$ and $F(x,y)=G(x)T(y)$. If, on the other hand, X and Y are not independent, then the probability generation function $F(1,y)$ need not agree with $T(y)$. One can check easily that

$$E_{RN} = \sum_k p_k k_{nn}(k) = F_y(1,1),$$

where F_y is the partial derivative of $F(x,y)$ with respect to y.

III. NETWORKS AND EVOLUTIONARY GAMES

Consider an evolutionary PD with payoffs as in Eq. (1). From the perspective of a defector, the cost of mutual cooperation is $T-R$. Additionally, the benefit paid out by a cooperator to a defector is $R-S$. Therefore, the PD has a cost-to-benefit ratio given by $\frac{T-R}{R-S}$. When game payoffs are normalized as in the discussion following Eq. (1), so that $P=S=0$, $R=1$, and $T=b$, it follows that the cost-to-benefit ratio is r, with $b=1+r$ and $0<r<1$.

Now suppose that agents occupy the vertices of the network. A single round of play consists of each agent engaging in a two-player PD with all of his immediate neighbors. During a round of play, agents maintain a pure strategy, exclusively playing one of either cooperate (C) or defect (D), in all interactions. Payoffs from each instance of the game accumulate through the round.

Following a round of play, the evolution is implemented using a discrete analog of the replicator dynamics [4,8]. Suppose that vertex v has accumulated a payoff of T_v during the round. Vertex v then updates his strategy by randomly choosing one from among all his neighbors for a payoff comparison. If vertex w with accumulated payoff T_w is chosen, then v adopts the strategy of w with probability $P_{v\to w}$, where

$$P_{v\to w} = \frac{\max\{0, T_w - T_v\}}{bk_{\max}} \quad (8)$$

and where $k_{\max}$ is the larger of the degrees of the vertices v and w.

Equation (8) is meant to mimic natural selection, where fit strategies are more likely to spread while less fit strategies die out. This widely studied updating rule [7,9,12,13,22,23] models strong selection, where fitness in the form of game payoffs is the principal driver of the evolution. For alternative models, including weak selection, see [35,36].

Simulations on a fixed network are carried out as follows. Define a *series* to be 10^4 rounds of playing and updating. The *series mean* is then the average fraction of cooperators over the last 1000 rounds of the series. One hundred series are run, each starting from a random initial configuration where the probability of an agent cooperating is 0.5. The equilibrium cooperation level on the network is taken to be the average of the 100 series means. For each network, the equilibrium cooperation level is computed for all values corresponding to a proper repeated PD, $1.05 \leq b < 2$, in the increment of 0.05.

In order to study the effects of network properties on evolutionary games, a diverse sample of networks—with varying heterogeneity and the presence and the absence of correlations—is required. To that end, consider a single parameter family of networks generated by an algorithm proposed in [37]. The algorithm interpolates between the Barabási-Albert (BA) model [30,31] and the Erdős-Rényi (ER) random graphs [38–40]. The BA model gives rise to heterogeneous networks, while the ER random graphs are essentially homogeneous. Each member of this family of networks will have N vertices, average degree of $2m$, and will be determined by a single scalable parameter α between 0 and 1.

The BA-ER family of networks is constructed by starting from a complete graph on n_0 vertices. A new vertex is chosen from the remaining set of all $N-n_0$ unconnected vertices. The new vertex has m edges to attach in the following way: with probability α, the vertex connects to any of the existing $N-1$ network vertices with a uniform probability. With probability $1-\alpha$, the edge attaches to an existing network vertex with probability proportional to the current degree of the vertex (i.e., by preferential attachment). The procedure is repeated m times for a particular vertex; that is, once for each edge.

When $\alpha=0$, the Barabási-Albert growth and preferential attachment algorithm of [30] is obtained, and the resulting network has a degree distribution that follows a power law, with $p_k \sim \frac{1}{k^3}$. When $\alpha=1$, an Erdős-Rényi random network with Poisson degree distribution is generated. For $0<\alpha<1$, the graph is a hybrid of the two with intermediate heterogeneity.

Using this algorithm, networks with 10^4 vertices are generated. For each choice of parameters $2m$ (average degree) and α, with $2m \in \{4,6\}$ and $\alpha \in \{0.00, 0.05, 0.10, 0.20, 0.30, 0.40, 0.60, 1.00\}$, four distinct networks are generated and labeled a, b, c, and d, respectively, from four independent instances of the algorithm. This gives a total of 64 graphs, denoted $\mathcal{B}_{2m,\alpha,l}$, with $2m$ and α as above, and $l \in \{a,b,c,d\}$.

Next, each $\mathcal{B}_{2m,\alpha,l}$ is distilled down to its degree distribution by throwing away all other specific contact information. A new graph is then reconstructed from the degree distribution using the configuration model of [34]. The result is a random network consistent with the specified degree distribution. This generates 64 additional networks denoted $\mathcal{CB}_{2m,\alpha,l}$ with $2m$, α, and l as above.

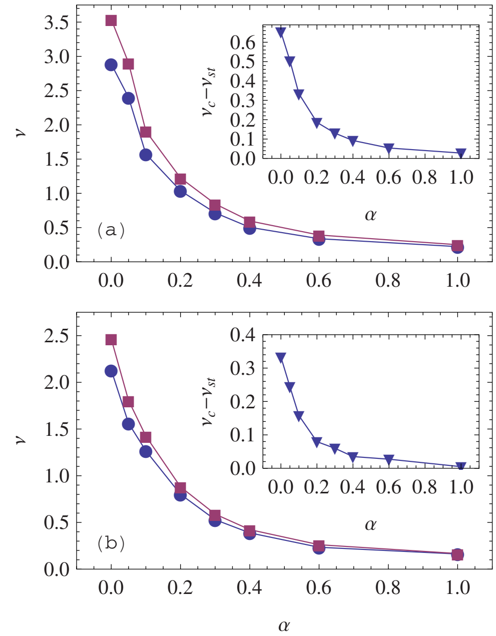

FIG. 1. (Color online) Plot of ν_c (squares) and ν_{st} (circles) for each value of α. Networks with an average degree of 4 are shown in (a) and those with an average degree of 6 are shown in (b). Each data point represents an average over all four networks with fixed α. Insets show, in each case, the difference $\nu_c-\nu_{st}$ as a function of α.

The final result, for each choice of parameter triple $2m,\alpha,l$, is a pair of networks $\mathcal{B}_{2m,\alpha,l}$ and $\mathcal{CB}_{2m,\alpha,l}$. The former has potential degree-degree correlations introduced by the growth and the preferential attachment component of the generating algorithm, while the latter is maximally random aside from its fixed degree distribution and has the property that correlations between vertex degrees at either end of a random edge are negligible. As a result, for $\alpha<1$, a network $\mathcal{B}_{2m,\alpha,l}$ will exhibit a nonconstant nearest-neighbor function k_{nn} and a correlated standard variance ν_c larger than the uncorrelated ν_{st} of its configuration model pair $\mathcal{CB}_{2m,\alpha,l}$ (whose k_{nn} function is essentially constant). This is summarized in Fig. 1.

Simulation results are given in Fig. 2 for the evolutionary PD described above on a sample of networks under consideration. As expected, cooperation is more successful on heterogeneous networks, and cooperation levels decrease more rapidly as a function of the game parameter b on homogeneous networks, which is consistent with results in [8,9,11–13,22]. Furthermore, cooperation is clearly enhanced by the presence of degree-degree correlations: cooperators perform better on the correlated realization of a fixed degree distribution than on the uncorrelated version of that same distribution, a phenomenon noticed in [9].

For example, consider the graphs $\mathcal{B}_{4,0,a}$ and $\mathcal{CB}_{4,0,a}$ in Fig. 2(a). Both share the same degree distribution and are therefore equally heterogeneous as measured by Eq. (3): $\nu_{st}=2.80$. However, cooperators are significantly more successful on the $\mathcal{B}_{4,0,a}$ network with degree-degree correlations. These correlations are evident in ν_c: using Eq. (7), $\mathcal{B}_{4,0,a}$ has $\nu_c=3.375$, while $\mathcal{B}_{4,0,a}$ has $\nu_c=\nu_{st}=2.80$. Similarly, Fig. 2(b) shows cooperators on networks with degree-degree correla-

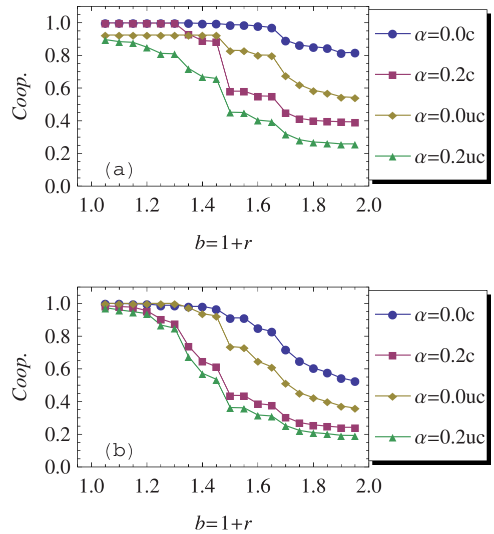

FIG. 2. (Color online) Equilibrium cooperation level as a function of the game parameter $b=1+r$. Panel (a) shows results on four networks with an average degree of 4: $\mathcal{B}_{4,0.0,a}$ and $\mathcal{B}_{4,0.2,a}$ have $\alpha=0.0$ and $\alpha=0.2$, respectively, and have degree-degree correlations (denoted $\alpha=0.0c$ and $\alpha=0.2c$ in the legend). Conversely, $\mathcal{CB}_{4,0.0,a}$ and $\mathcal{CB}_{4,0.2,a}$ also have $\alpha=0.0$ and $\alpha=0.2$ but are uncorrelated (denoted $\alpha=0.0uc$ and $\alpha=0.2uc$ in the legend). Panel (b) gives results for the corresponding networks with an average degree of 6.

tions outperforming cooperators on uncorrelated networks. Note that even small-scale correlations give a notable boost to cooperators. For instance, considering $\mathcal{B}_{6,0.2,a}$ and $\mathcal{CB}_{6,0.2,a}$, $\nu_c=0.902$ while $\nu_{st}=0.821$, for a difference of only 0.081. However, Fig. 2(b) clearly shows that cooperation is more successful on $\mathcal{B}_{6,0.2,a}$.

To obtain a single numerical measure of the success of cooperators in the evolutionary PD on a network, cooperation levels are averaged over the game parameters [21]. Specifically, write $b=1+r$, where r is the cost-to-benefit ratio of cooperation and $0.05\leq r<1$. Let $c_{\mathcal{N},r}$ be the equilibrium cooperation level in the evolutionary PD on $\mathcal{N}$ with game parameter $b=1+r$. The global average cooperation $\bar{c}_{\mathcal{N}}$ on $\mathcal{N}$ is then defined to be the weighted average

$$\bar{c}_{\mathcal{N}}=\frac{\sum\limits_{0.05\leq r<1} rc_{\mathcal{N},r}}{\sum\limits_{0.05\leq r<1} r},$$

with sums taken over $0.05\leq r<1$ in the increment of 0.05. Weighing $c_{\mathcal{N},r}$ by r amounts to rewarding a network more when it can sustain cooperation despite hosting a game that is inherently hostile to cooperators. Further, the cost-to-benefit ratio is a natural choice of weight since games that differ by a constant factor in the payoffs will share the same cost-to-benefit ratio r. Moreover, regardless of the choice of payoffs consistent with the repeated PD, the cost-to-benefit ratio gives a natural parameter in the interval (0,1) associated with the game. However, r is simply a choice of weight, and alternative choices such as b, or no weight at all, give similar results in what follows as well as in [21]. We note that the regression line in [21] is most natural, however, in the sense that the absolute values of the coefficients in the linear regression are closest to 1 when r is used as the weight.

Before proceeding, we recall previous studies on the role played by degree-degree correlations in the evolutionary PD. In [41], this question was addressed in the context of scale-free networks. The starting point in that study was a BA network with a small Newman correlation coefficient (NCC) [42]. From there, the network was subjected to a reshuffling algorithm that decreased (increased) the NCC by linking big vertices to other small (big) vertices, and the effects on cooperation were documented. However, as pointed out in [23] and shown above, BA networks are not uncorrelated despite having small NCC [the nearest-neighbor function $k_{nn}(k)$ is not constant]. Thus, the reshuffling algorithm adds degree-degree correlations to an already correlated network, making it difficult to isolate the effects of correlations on cooperation. This issue was taken up in [23] and further studied there in the context of networks with monotonic nearest-neighbor functions, but the question of correlation effects in networks generated by growth and preferential attachment mechanisms was left open. In the following, we clarify the role played by degree-degree correlations in this case and, more generally, by quantifying their effect on network heterogeneity and interpreting that role as a local mechanism involving agents and their neighbors as in [21].

IV. RESULTS AND DISCUSSION

Each network $\mathcal{N}$ has an associated ordered pair $(\nu_{c,\mathcal{N}},\bar{c}_{\mathcal{N}})$. In Fig. 3(a) [Fig. 3(b)] these ordered pairs are plotted for the $\mathcal{CB}_{2m,\alpha,l}$ [$\mathcal{B}_{2m,\alpha,l}$] networks, for all possible triples $2m,\alpha,l$. Both plots show a strong positive correlation between $\nu_{c,\mathcal{N}}$ and $\bar{c}_{\mathcal{N}}$: increased heterogeneity, quantified by ν_c, leads to an increased cooperation [43]. However, when the two plots are combined in Fig. 3(c), cooperation levels seen on correlated networks are significantly higher than those expected on uncorrelated networks with the same heterogeneity. This discrepancy is explored below.

The precise relationship between average cooperation and standardized variance ν_{st} in the case of the uncorrelated $\mathcal{CB}_{2m,\alpha}$ networks [Fig. 3(a)] was determined in [21,44]. Recall that an uncorrelated network's epidemic threshold [26,27,33] is defined to be

$$\lambda=\frac{\langle k\rangle}{\langle k^2\rangle}.$$

Intuitively, an epidemic outbreak of a disease is possible if the probability the disease propagates along a contact edge from an infected agent to a susceptible agent is larger than λ. Equation (2) implies that $\lambda=\frac{1}{T'(1)}$. Let $\mu_{st}=\langle k\rangle\lambda$. That is, μ_{st} is the network's epidemic threshold scaled by its average degree. It follows from Eq. (3) that

$$\mu_{st}=\frac{G'(1)}{T'(1)}=\frac{1}{1+\nu_{st}}. \quad (9)$$

This connection between evolutionary games on networks and epidemic outbreaks has been discussed, in different contexts, in [45,46].

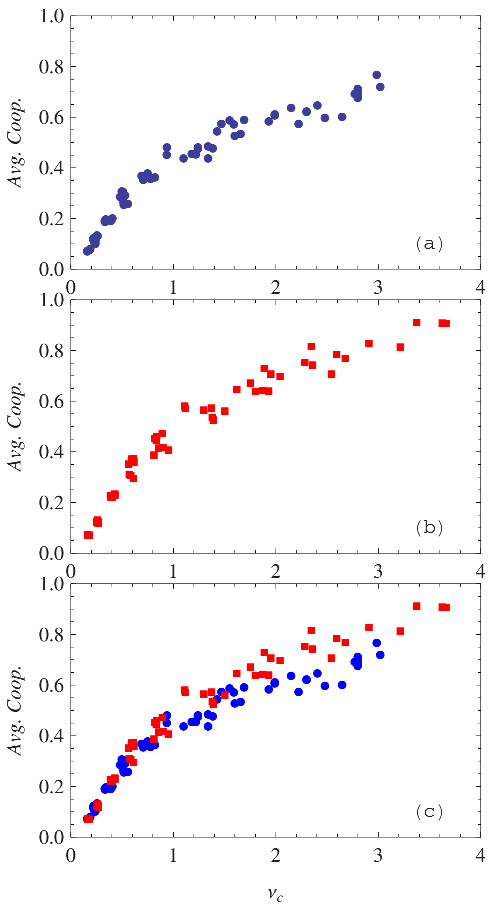

FIG. 3. (Color online) Cooperation versus ν_c in (a) the uncorrelated and the (b) correlated cases. Panel (c) shows (a) and (b) simultaneously.

Figure 4 (circles) show $\bar{c}_{\mathcal{N}}$ plotted as a function of μ_{st} for the uncorrelated $\mathcal{CB}_{2m,\alpha}$ networks and recovers the results of [21]: there is a strong linear correlation between global average cooperation and μ_{st}. Using an analogy whereby cooperation (or defection) propagates on the network like a disease, and considering that the susceptibility of an agent depends on his fitness relative to the fitness of his neighbor as in Eq. (8), the dependence of cooperation on the expression in Eq. (9) is intuitive. The network's epidemic threshold

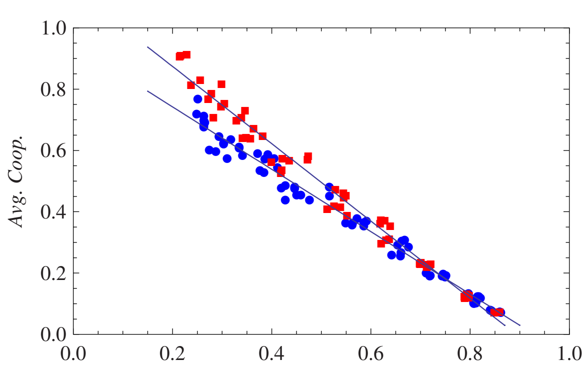

FIG. 4. (Color online) Cooperation versus μ_c in the uncorrelated (circles) and correlated (squares) cases.

λ is the reciprocal of average degree of a random neighbor and, when scaled by the average degree of a random vertex $\langle k\rangle$, predicts the success of cooperation on the network.

It is natural, therefore, to consider the analogous situation on the correlated $\mathcal{B}_{2m,\alpha}$ networks. Replacing ν_{st} and $T'(1)$ with their correlated counterparts ν_c and E_{RN}, define

$$\mu_c = \frac{\langle k\rangle}{E_{RN}} = \frac{1}{1+\nu_c}. \tag{10}$$

Of course, if the network is uncorrelated then $\mu_c=\mu_{st}$. Figure 4 (squares) shows a plot of average cooperation as a function of μ_c for the correlated $\mathcal{B}_{2m,\alpha}$ networks. Just as in the uncorrelated case, there is a linear dependence of average cooperation on μ_c. The fit of the data to the correlated regression line is quite strong with $r^2=0.982$ (compared with $r^2=0.985$ for the regression on the uncorrelated network data).

Given the interpretation of $\frac{\mu_{st}}{\langle k\rangle}$ as the epidemic threshold of an uncorrelated random network, and the similarity in the behavior of the evolutionary PD dynamics in the uncorrelated and correlated cases, $\frac{\mu_c}{\langle k\rangle}$ can be thought of as a candidate for a generalized notion of an epidemic threshold of a correlated network.

Figure 4 also reveals more clearly the dissonance between the correlated and the uncorrelated cases seen in Fig. 3(c). While average cooperation is indeed linear in μ_c (μ_{st}) on the correlated (uncorrelated) networks, the data sets for the correlated networks and the uncorrelated networks lie on distinct regression lines.

For large μ_c, and therefore small ν_c, the average cooperation levels seen on the two families of networks are quite similar. These networks correspond to larger values of α and are closer to the ER random graph model where degree-degree correlations are minimal. Therefore, the effects of degree-degree correlations on cooperation on these networks are small. Small values of μ_c, and therefore large ν_c, correspond to networks with smaller α for which the preferential attachment algorithm of the BA model is prominent. This leads to significant degree-degree correlations and a disassortative network where smaller vertices tend to have larger neighbors. While μ_c measures the increased heterogeneity, interpreted as size difference between an average vertex and his neighbor, due to these correlations, Fig. 4 shows that the correlations provide an additional enhancement to cooperation beyond that expected if μ_c (and therefore network heterogeneity as defined above) were the sole indicator of network cooperation. Network heterogeneity coming from degree-degree correlations has a disproportionate effect on network cooperation.

Consider, therefore, two distinct mechanisms contributing to the success of cooperation on a network. The first is a functional form of the standardized variance of the degree distribution, namely, $\mu_{st}=\frac{G'(1)}{T'(1)}$ and provides a base-line level of success for cooperation on a network [21]. The second is the increase in network heterogeneity contributed by degree-degree correlations, quantified by $\Delta=\mu_{st}-\mu_c=\frac{G'(1)}{T'(1)}-\frac{G'(1)}{E_{RN}}$, and provides a correction to this base line.

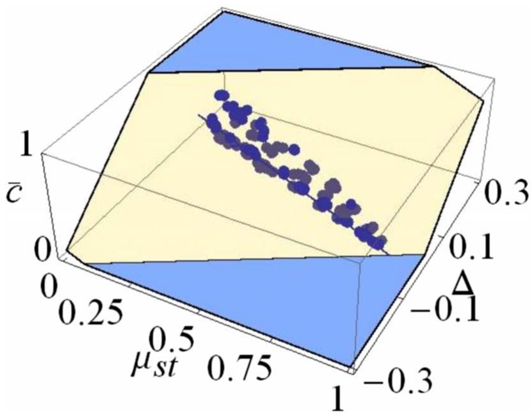

FIG. 5. (Color online) The plane determined by the two-variable linear regression plotted with the full data set. The regression equation is $\bar{c}=0.982+3.006\Delta-1.088\mu_{st}$, with linear correlation coefficient $r^2=0.979$. The darker shading at the upper and the lower corners of the plot indicate that the plane is outside the viewing box. The line shows the linear regression in [21], which provides the base-line cooperation for data collected on uncorrelated networks. The figure shows average cooperation increasing along the plane with increased degree-degree correlations quantified by Δ.

Figure 5 shows the full data set, as well as the plane resulting from a two-variable linear regression on Δ and μ_{st}. The regression equation is

$$\bar{c} = 0.982 + 3.006\Delta - 1.088\mu_{st}. \tag{11}$$

The uncorrelated data lies along the original regression line given in [21] and shown in Fig. 4 (circles) and is obtained by taking $\Delta=0$ in Eq. (11). The line is pictured in Fig. 5 as well. Cooperation is at its lowest for this data set, affirming the notion of μ_{st} giving a base line for network cooperation.

For $\Delta \neq 0$, the marginal benefit to cooperators coming from degree-degree correlations moves the cooperation level up along the plane as determined by Eq. (11). The regression line fitting the squares in Fig. 4 is also visible in Fig. 5 and lies on the plane given in Eq. (11). The fit of the data set to the regression plane in Eq. (11) is excellent, with $r^2=0.979$.

A closer look at Δ provides additional perspective

$$\Delta = \mu_{st} - \mu_c = \frac{G'(1)}{T'(1)} - \frac{G'(1)}{E_{RN}} = \frac{G'(1)[E_{RN} - T'(1)]}{T'(1)E_{RN}} = \frac{G'(1)}{E_{RN}}\left[\frac{E_{RN} - T'(1)}{T'(1)}\right].$$

That is, Δ, the contribution of degree-degree correlations to the heterogeneity of the network, can be thought of as the product of two terms. The first term is intrinsic to the network: the ratio of average degree of a randomly chosen vertex to average degree of a randomly chosen neighbor. The second term, $\frac{E_{RN}-T'(1)}{T'(1)}$, is extrinsic. It measures the distance, in the sense of average degree of a neighbor, between the network and a generic random network sharing the same degree distribution [cf. Eq. (3)]. The generic random network case is achieved, for example, using the configuration model, where $E_{RN}=T'(1)$. Figure 5 shows that Δ positively contributes to the success of cooperation on the network and Eq. (11) quantifies that contribution.

V. CONCLUSIONS

In this paper, we have generalized methods used in [21] to give analytical measures of heterogeneity for networks with degree-degree correlations such as Barabási-Albert scale-free networks generated via growth and preferential attachment. Using these methods, we studied an evolutionary prisoner's dilemma on correlated networks and found two appropriate parameters that measure network heterogeneity.

The first parameter is a base-line heterogeneity measure due to the underlying degree distribution of the network. This parameter is given by a functional form of standardized variance and is related to the epidemic threshold in the uncorrelated network case. This could give insight into an appropriate measure of the epidemic threshold on correlated networks. The second parameter is a measure of the added contribution to network heterogeneity coming from degree-degree correlations, as compared to a generic uncorrelated network with the same underlying degree distribution.

Extending the results in [21], we have shown the existence of a strong linear correlation between these two parameters and the average cooperation level in an evolutionary prisoner's dilemma on a network. These results help to both quantify and clarify the influence of degree-degree correlations on cooperation phenomena in an evolutionary PD on a network.

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