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Renormalized Path Integral for the Two-dimensional δ-function Interaction

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A path-integral approach for δ-function potentials is presented. Particular attention is paid to the two-dimensional case, which illustrates the realization of a quantum anomaly for a scale-invariant problem in quantum mechanics. Our treatment is based on an infinite summation of perturbation theory that captures the nonperturbative nature of the δ-function bound state. The well-known singular character of the two-dimensional δ-function potential is dealt with by considering the renormalized path integral resulting from a variety of schemes: dimensional, momentum-cutoff, and real-space regularization. Moreover, compatibility of the bound-state and scattering sectors is shown.

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I. INTRODUCTION

It is well known that the study of singular potentials in quantum mechanics leads to an array of technical difficulties [1]. In the past few years the relevance of field-theory tools for the analysis of singular potentials has been gradually recognized and successfully applied to a number of interesting cases. However, most of the published research to date has been performed in the Schrödinger picture by directly renormalizing the solutions obtained from the Schrödinger equation. In contrast with these earlier works, we have recently advanced a broad program that includes as a major objective the treatment of these potentials within a path-integral framework [2]. The successful completion of this goal should provide the first natural step in our program, the ultimate purpose of which is to tackle the notoriously difficult problem of bound states in quantum field theory [3,4]. The relevance of singular potentials is highlighted by their inevitable presence in the nonrelativistic reduction from field theory to quantum mechanics [5,6] within the effective-field-theory paradigm [7]. In particular, in Ref. [2] we were able to show explicitly that: (i) *infinite summations of perturbation theory* capture the intrinsic nonperturbative nature of bound states [8,9]; (ii) the use of this technique and a *renormalization analysis* [10–13] provide a path-integral rederivation of the solution to the inverse square potential [14]. In this paper we now show that a similar approach reproduces the behavior of the two-dimensional δ-function potential, another apparently pathological case [15–17]. A central result of this research is the *predictive power* of the general framework: once renormalization is implemented at the level of the bound state(s) of the theory, the scattering observables are uniquely determined.

The two-dimensional δ-function interaction is actually included in a larger class of singular potentials whose phenomenological usefulness dates back to the introduction of pseudopotentials in the early days of quantum mechanics [18], and the subsequent applications of the zero-range potential in nuclear physics [19,20], condensed-matter physics [21], statistical mechanics [22], atomic physics [23], and particle physics [15]. Our understanding of these singular problems has advanced considerably in recent times [24]: with the modern theory of pseudopotentials [25,26], which overlaps with the technique of self-adjoint extensions [17,27,28]; with the study of the nonrelativistic limit of the ϕ^4 theory and the question of its triviality [29] and with the application of quantum-field-theory tools to singular problems in quantum mechanics. In particular, there have been many renormalization analyses of the two-dimensional δ-function potential: using momentum-space regularization [15–17,30–33], as well as a momentum-space renormalization-group analysis [34]; using real-space regularization [35–38]; using dimensional regularization [12,13,30,31]; and dealing with its associated quantum anomaly [17,39,40].

In contrast with the research published to date, in this paper we deal with the renormalization of the two-dimensional δ-function potential by using path integrals and infinite summations of perturbation theory from the outset. In Sec. II we introduce the general framework, followed by a presentation of the δ-function interaction in Sec. III. Renormalization of the bound-state sector is analyzed within dimensional regularization in Sec. IV, momentum-cutoff regularization in Sec. V, and real-space regularization in Sec. VI. This is followed by an analysis of the scattering sector in Sec. VII and conclusions in Sec. VIII. Finally, the appendices summarize the derivation and properties of the free-particle Green's functions used throughout the paper.

II. GENERAL FRAMEWORK: INFINITE SUMMATIONS OF PERTURBATION THEORY

Our stated goal of renormalizing the δ-function interaction is best accomplished by first establishing the general framework. We will now focus on the general technique of infinite summations of perturbation theory and start by introducing the path-integral framework in its different formulations needed for subsequent derivations.

A. General definitions

The central object in the path-integral treatment for a nonrelativistic particle of mass M is the quantum-mechanical propagator

$$K_D(\mathbf{r}'',\mathbf{r}';t'',t')=\int_{\mathbf{r}(t')=\mathbf{r}'}^{\mathbf{r}(t'')=\mathbf{r}''}\mathcal{D}\mathbf{r}(t)\times\exp\left\{\frac{i}{\hbar}S[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')\right\} \tag{1}$$

$$=\lim_{N\to\infty}\left(\frac{M}{2\pi i\epsilon\hbar}\right)^{DN/2}\left[\prod_{k=1}^{N-1}\int_{\mathbb{R}^D}d^D\mathbf{r}_k\right]\times\exp\left\{\frac{i}{\hbar}\sum_{j=0}^{N-1}\left[\frac{M(\mathbf{r}_{j+1}-\mathbf{r}_j)^2}{2\epsilon}-\epsilon V(\mathbf{r}_j,t_j)\right]\right\}, \tag{2}$$

where the interaction with a potential $V(\mathbf{r},t)$ is considered in the generic D-dimensional case. In the continuous version, Eq. (1), $S[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')$ stands for the classical action associated with paths $\mathbf{r}(t)$ connecting the end points $\mathbf{r}(t')=\mathbf{r}'$ and $\mathbf{r}(t'')=\mathbf{r}''$. In the time-lattice version, Eq. (2), a time slicing

$$t_j=t'+j\epsilon,\quad \epsilon=\frac{(t''-t')}{N}\quad (j=0,\ldots,N) \tag{3}$$

is introduced, with the additional notation $\mathbf{r}_j=\mathbf{r}(t_j)$, while the end points satisfy $t_0\equiv t'$, $t_N\equiv t''$, $\mathbf{r}_0\equiv\mathbf{r}'$, and $\mathbf{r}_N\equiv\mathbf{r}''$. In Ref. [2] we showed how to derive the general expressions for the required infinite summations by starting with the basic lattice definition (2); these results are reviewed below.

For the particular case of a time-independent potential—which applies to the δ-function interactions of this paper—Eq. (1) depends upon the times t' and t'' of the end points only through their difference $T=t''-t'$, so that we write $K_D(\mathbf{r}'',\mathbf{r}';t'',t')\equiv K_D(\mathbf{r}'',\mathbf{r}';T)$. Then, a complete analysis of the spectrum is most effectively carried out by introducing the energy Green's function $G_D(\mathbf{r}'',\mathbf{r}';E)$ as the Fourier transform with respect to T of the retarded Green's function $G_D(\mathbf{r}'',\mathbf{r}';T)=K_D(\mathbf{r}'',\mathbf{r}';T)\,\theta(T)$, where $\theta(T)$ stands for the Heaviside function. Explicitly [2,41],

$$G_D(\mathbf{r}'',\mathbf{r}';E)=\frac{1}{i\hbar}\int_0^\infty dT e^{iET/\hbar}K_D(\mathbf{r}'',\mathbf{r}';T) \tag{4}$$

$$=\frac{1}{i\hbar}\int_0^\infty dT\int_{\mathbf{r}(0)=\mathbf{r}'}^{\mathbf{r}(T)=\mathbf{r}''}\mathcal{D}\mathbf{r}(t)\times\exp\left(\frac{i}{\hbar}\{S[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';T)+ET\}\right) \tag{5}$$

is defined so that

$$G_D(\mathbf{r}'',\mathbf{r}';E)=\langle\mathbf{r}''|(E-\hat{H}+i\delta)^{-1}|\mathbf{r}'\rangle, \tag{6}$$

which is the usual definition in terms of the Green's function operator, with $\delta=0^+$.

B. Perturbation theory and infinite summations

A thorough analysis of the problem at hand can be completed by means of perturbative expansions, as we shall discuss next. At first sight one might dismiss this approach because it is well known that finite summations of perturbation theory are unable to capture the required nonperturbative behavior. However, this limitation can be removed when infinite summations are considered. The technique of infinite perturbative summations [8] has already been successfully applied to a number of interesting problems [2,9,42]. Let us now summarize the main results of this approach, as needed for the present paper.

The perturbation expansion starts with the resolution of the action

$$\begin{aligned}S[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')&=S^{(0)}[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')+\tilde{S}[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')\\&=S^{(0)}[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')-\int_{t'}^{t''}dt V(\mathbf{r}(t),t),\end{aligned} \tag{7}$$

where $S^{(0)}[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')$ is the action for the "unperturbed" problem whose solution is already known, while $\tilde{S}[\mathbf{r}(t)](\mathbf{r}'',\mathbf{r}';t'',t')$ represents the perturbation in the form of additional interactions $V(\mathbf{r},t)$. In what follows, these interactions and the unperturbed Hamiltonian will be assumed to be time independent—the generalization for time-dependent potentials is straightforward and discussed in Ref. [2] and will not be pursued in this paper. The expansions are generated through the Taylor series of $e^{i\tilde{S}/\hbar}$ in Eq. (1), combined with a proper rearrangement of the time lattice that defines the path integral [2,43]. This procedure leads to the following infinite perturbative series for the propagator:

$$K_D(\mathbf{r}'',\mathbf{r}';T)=\sum_{n=0}^{\infty}\prod_{\alpha=1}^{n}\left[\int d^D\mathbf{r}_\alpha\frac{V(\mathbf{r}_\alpha)}{i\hbar}\right]\times\int_{t'}^{t''}dt_n\int_{t'}^{t_n}dt_{n-1}\cdots\int_{t'}^{t_2}dt_1\times\prod_{\beta=0}^{n}[K_D^{(0)}(\mathbf{r}_{\beta+1},\mathbf{r}_\beta;t_{\beta+1}-t_\beta)], \tag{8}$$

where it is implicitly understood that, at every order n, $t_0\equiv t'$, $t_{n+1}\equiv t''=t'+T$, $\mathbf{r}_0\equiv\mathbf{r}'$, and $\mathbf{r}_{n+1}\equiv\mathbf{r}''$; its corresponding energy Green's function becomes

$$G_D(\mathbf{r}'',\mathbf{r}';E)=\sum_{n=0}^{\infty}\prod_{\alpha=1}^{n}\left[\int_0^\infty d^D\mathbf{r}_\alpha V(\mathbf{r}_\alpha)\right]\times\prod_{\beta=0}^{n}[G_D^{(0)}(\mathbf{r}_{\beta+1},\mathbf{r}_\beta;E)]. \tag{9}$$

In Eqs. (8) and (9), $K_D^{(0)}(\mathbf{r}'',\mathbf{r}';T)$ and $G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E)$ stand for the unperturbed propagator and energy Green's function, respectively. In particular, when the unperturbed problem is the free-particle case, as we will assume for the remainder of this paper, the results of Appendix A [specifically Eq. (A4)] imply that

$$G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E_k)=\frac{2M}{\hbar^2}\mathcal{G}_D^{(+)}(\mathbf{R};k), \tag{10}$$

where

$$k=\frac{\sqrt{2ME_k}}{\hbar}, \quad \mathbf{R}=\mathbf{r}-\mathbf{r}', \tag{11}$$

and the rescaled Green's function $\mathcal{G}_D^{(+)}(\mathbf{R};k)$ is explicitly given by the closed analytical expression

$$\mathcal{G}_D^{(+)}(\mathbf{R};k)=-\frac{i}{4}\left(\frac{k}{2\pi R}\right)^{\nu}H_\nu^{(1)}(kR), \tag{12}$$

with

$$\nu=\frac{D}{2}-1, \tag{13}$$

and $H_\nu^{(1)}(\xi)$ being the Hankel function of the first kind and order ν. As shown in Appendix B, Eq. (B7), the function (12) is identical to the D-dimensional free-particle causal energy Green's function (with outgoing boundary conditions), in agreement with the general result (6). Finally, in the analysis of the bound-state sector of the theory, the analytic continuation [cf. Eqs. (B5) and (B6)]

$$\mathcal{K}_D(\mathbf{R};\kappa)=\mathcal{G}_D^{(+)}(\mathbf{R};k=i\kappa)=-\frac{1}{2\pi}\left(\frac{\kappa}{2\pi R}\right)^{\nu}K_\nu(\kappa R) \tag{14}$$

will become ubiquitous.

C. Central potentials

For interactions with central symmetry in D dimensions, an alternative approach is to rewrite the propagator in hyperspherical polar coordinates [2,12,44], in terms of which Eq. (1) admits the expansion [45]

$$K_D(\mathbf{r}'',\mathbf{r}';T)=(r''r')^{-(D-1)/2}\sum_{l=0}^{\infty}\sum_{m=1}^{d_l}Y_{lm}(\mathbf{\Omega}'')\times Y_{lm}^{*}(\mathbf{\Omega}')K_{l+\nu}(r'',r';T), \tag{15}$$

where ν is defined in Eq. (13), $Y_{lm}(\mathbf{\Omega})$ stands for the hyperspherical harmonics, and $d_l=(2l+D-2)(l+D-3)!/l!(D-2)!$ [44]. This is established by introducing the necessary hyperspherical angular coordinates and repeatedly using the addition formula [45] $e^{iz\cos\psi}=(iz/2)^{-\nu}\Gamma(\nu)\Sigma_{l=0}^{\infty}(l+\nu)I_{l+\nu}(iz)C_l^{(\nu)}(\cos\psi)$, where $I_p(z)$ stands for the modified Bessel function of the first kind, while $C_l^{(\nu)}(x)$ is a Gegenbauer polynomial [46]. Equation (15) introduces the radial propagator

$$\begin{aligned}K_{l+\nu}(r'',r';T)&=\lim_{N\to\infty}\left(\frac{M}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\left[\int_0^{\infty}dr_k\right]\mu_{l+\nu}^{(N)}[r^2]\\&\quad\times\exp\left\{\frac{i}{\hbar}\sum_{j=0}^{N-1}\left[\frac{M}{2\epsilon}(r_{j+1}-r_j)^2-\epsilon V(r_j)\right]\right\}\\&=\int\mathcal{D}r(t)\mu_{l+\nu}[r^2]\exp\left(\frac{i}{\hbar}\int_{t'}^{t''}dt\right.\\&\quad\left.\times\left\{\frac{M}{2}[\dot{r}(t)]^2-V(r(t))\right\}\right)\end{aligned} \tag{16}$$

for each angular momentum channel l. Here a radial functional weight $\mu_{l+\nu}^{(N)}[r^2]$ has been properly defined [47] so that the radial path integral, supplemented by the condition $r(t)\geqslant 0$, can be given a formal continuum representation in terms of the usual one-dimensional path-integral pseudomeasure $\mathcal{D}r(t)$; explicitly,

$$\mu_{l+\nu}^{(N)}[r^2]=\prod_{j=0}^{N-1}\left[\sqrt{2\pi z_j}e^{-z_j}I_{l+\nu}(z_j)\right], \tag{17}$$

where $z_j=Mr_jr_{j+1}/i\epsilon\hbar$. The perturbation expansion for the radial propagator follows by directly resolving Eq. (8) into its partial-wave components,

$$\begin{aligned}K_{l+\nu}(r'',r';T)&=\sum_{n=0}^{\infty}\prod_{\alpha=1}^{n}\left[\int_0^{\infty}dr_\alpha\frac{V(r_\alpha)}{i\hbar}\right]\\&\quad\times\int_{t'}^{t''}dt_n\int_{t'}^{t_n}dt_{n-1}\cdots\int_{t'}^{t_2}dt_1\\&\quad\times\prod_{\beta=0}^{n}\left[K_{l+\nu}^{(0)}(r_{\beta+1},r_\beta;t_{\beta+1}-t_\beta)\right].\end{aligned} \tag{18}$$

Similarly, by performing a Fourier transform of Eq. (18) or by resolving Eq. (9) in partial waves, the radial energy Green's function is given by

$$G_{l+\nu}(r'',r';E)=\sum_{n=0}^{\infty}\prod_{\alpha=1}^{n}\left[\int_0^{\infty}dr_\alpha V(r_\alpha)\right]\times\prod_{\beta=0}^{n}\left[G_{l+\nu}^{(0)}(r_{\beta+1},r_\beta;E)\right]. \tag{19}$$

In particular, if the starting (unperturbed) problem is the free particle, as reviewed in Appendix A [from Eq. (A8)], the required radial Green's function becomes

$$G_{l+\nu}^{(0)}(r'',r';E)=-\frac{2M}{\hbar^2}\sqrt{r'r''}I_{l+\nu}(\kappa r_<)K_{l+\nu}(\kappa r_>), \tag{20}$$

where $K_p(x)$ is the modified Bessel function of the second kind and order p (Macdonald function) [46], while $r_<(r_>)$ is the smaller (larger) of r' and r''.

III. δ-FUNCTION INTERACTIONS

We are now ready to start applying the general framework to δ-function potentials. A D-dimensional δ-function interaction centered at the origin is represented by the potential

$$V(\mathbf{r})=\sigma\delta^{(D)}(\mathbf{r}), \tag{21}$$

where the coupling strength will be conveniently rewritten as

$$\sigma=-\frac{\hbar^2}{2M}\lambda, \tag{22}$$

a replacement that will simplify the ensuing dimensional expressions; in particular $[\lambda]=1$ (dimensionless) for $D=2$.

Many of the equations to be shown in this section have already been derived in a number of different contexts. In particular, Eq. (8) can be laboriously evaluated by using a recurrence relation and its particular expression for $D=1$ agrees with the result known in the literature [48]. On the other hand, in this paper we will focus on the expansion of the energy Green's function (9), which can be summed as a geometric series. In effect, rewriting Eq. (9) term by term,

$$G_D(\mathbf{r}'',\mathbf{r}';E)=\sum_{n=0}^{\infty}G_D^{(n)}(\mathbf{r}'',\mathbf{r}';E), \tag{23}$$

each factor $\int_0^\infty d^D\mathbf{r}_\alpha V(\mathbf{r}_\alpha)$ is merely reduced to σ and carries the additional instruction that the replacement $\mathbf{r}_\alpha\to\mathbf{0}$ be made. Then,

$$G_D^{(n)}(\mathbf{r}'',\mathbf{r}';E)=\sigma^n[G_D^{(0)}(\mathbf{0},\mathbf{0};E)]^{n-1}\times G_D^{(0)}(\mathbf{r}'',\mathbf{0};E)G_D^{(0)}(\mathbf{0},\mathbf{r}';E), \tag{24}$$

for $n\geqslant 1$, while the term of order zero maintains its value $G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E)$. Accordingly, the exact infinite summation of this series for finite σ leads to the expression [42]

$$G_D(\mathbf{r}'',\mathbf{r}';E)=G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E)-\frac{G_D^{(0)}(\mathbf{r}'',\mathbf{0};E)G_D^{(0)}(\mathbf{0},\mathbf{r}';E)}{G_D^{(0)}(\mathbf{0},\mathbf{0};E)-1/\sigma}. \tag{25}$$

Remarkably, Eq. (25) summarizes the complete physics of a δ-function perturbation applied to any problem whose action is $S^{(0)}$ and exactly described by $G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E)$.

There is only one apparent restriction in the above derivation: the geometric series involved in the infinite perturbation expansion is guaranteed to converge only for $|\sigma|<|G_D^{(0)}(\mathbf{0},\mathbf{0};E)|^{-1}$; however, this restriction can be lifted by noticing that the final expression (25) provides the desired analytic continuation in the complex λ plane.

In this paper we will consider the particular case when $S^{(0)}$ represents a free-particle action. Furthermore, Eq. (25) will now be used to study the bound-state sector of the theory, while in Sec. VII we will reformulate it in the language of the T matrix for the scattering sector. The bound states are immediately recognized from the pole(s) of Eq. (25); explicitly, with the rescaled Green's function $\mathcal{G}_D^{(+)}(\mathbf{R};k)$ [Eq. (10)], it follows that

$$\mathcal{G}_D^{(+)}(\mathbf{0};k=i\kappa)=-\frac{1}{\lambda}, \tag{26}$$

a condition that implies the existence of at most one bound state. This bound-state equation can be analyzed in terms of the behavior of the D-dimensional Green's function (12) near the origin. In particular, Eq. (26) displays a unique pole when $\mathcal{G}_D^{(+)}(\mathbf{0};k)$ is finite, in which case the ground-state wave function—derived from the corresponding residue—becomes [from Eqs. (14) and (25)]

$$\Psi_{(\mathrm{gs})}(\mathbf{r})\propto G_D^{(0)}(\mathbf{0},\mathbf{r};E_k)|_{k=i\kappa}\propto\mathcal{G}_D^{(+)}(\mathbf{r};k=i\kappa)\propto r^{-\nu}K_\nu(\kappa r), \tag{27}$$

where κ is the solution to Eq. (26) and an appropriate normalization constant should be introduced. Let us now see the details of this technique for particular dimensionalities.

For the one-dimensional case, described by a coordinate $x\in(-\infty,\infty)$, the radial variable becomes $R=|x|$, while the Green's functions can be rewritten in terms of $H_{-1/2}^{(1)}(\xi)=(2/\pi\xi)^{1/2}e^{i\xi}$ and $K_{-1/2}(\xi)=(\pi/2\xi)^{1/2}e^{-\xi}$. Then, Eqs. (12) and (26) provide the condition $\kappa=\lambda/2$, which implies a ground-state energy $E=-(\hbar\lambda/2)^2/2M$ and ground-state wave function $\Psi_{(\mathrm{gs})}(x)=\sqrt{\kappa}e^{-\kappa|x|}$, in agreement with the known textbook answers [48].

We will now focus on the two-dimensional case, which exhibits a number of peculiar features. For the two-dimensional δ-function interaction, Eqs. (12) and (26) lead to a divergent expression and regularization is called for. This is the problem we announced earlier and to which we now turn our attention.

IV. DIMENSIONAL REGULARIZATION

In dimensional regularization [49,50], one generalizes the expressions from a given physical dimensionality D_0 to $D=D_0-\epsilon$, with $\epsilon=0^+$. In quantum mechanics, this procedure is implemented by analytically extending the potential $V(\mathbf{r})\equiv V^{(D_0)}(\mathbf{r})$ from D_0 to D dimensions, where it takes a generalized form $V^{(D)}(\mathbf{r})$. The goal of this construct is to reformulate the problem in terms of a regularized potential $V^{(D)}(\mathbf{r})$ that is no longer singular. Even though this generalization is somewhat arbitrary, the requirement that it be regular suggests the following straightforward definition for $V^{(D)}(\mathbf{r})$ [12]:

$$\begin{tikzcd}
 & \text{real space} & & \text{reciprocal space} \\
D_0 \text{ dimensions:} & V(\mathbf{r})=V^{(D_0)}(\mathbf{r}) \arrow[rr, "\mathcal{F}_{(D_0)}"] \arrow[d, "\mathcal{D}_{D_0\to D}"'] & & \tilde{V}(\mathbf{k})=\tilde{V}^{(D_0)}(\mathbf{k}) \arrow[d, "\mathcal{D}_{D_0\to D}"'] \\
D \text{ dimensions:} & V^{(D)}(\mathbf{r}) & & \tilde{V}^{(D)}(\mathbf{k})=\tilde{V}(\mathbf{k}) \arrow[ll, "\mathcal{F}^{-1}_{(D)}"']
\end{tikzcd}, \tag{28}$$

where, in this commutative diagram, $\mathcal{F}_{(D_0)}$ is the Fourier transform in D_0 dimensions, $\mathcal{F}_{(D)}^{-1}$ the inverse Fourier transform in D dimensions, and $\mathcal{D}_{D_0\to D}$ is a shorthand for dimensional continuation from D_0 to D dimensions. This procedure works because the homogeneous property of Fourier transforms guarantees that if the potential has a singular homogeneous behavior of degree -2, then its counterpart $V^{(D)}(\mathbf{r})$ is homogeneous of degree $-2+\epsilon$, and therefore regular [12]. For example, if one attempted a solution of the inverse square potential, naive generalizations (such as keeping a $1/r^2$ potential and simply changing the dimensionality) would fail, but the prescription (28) would be successful [12,13].

At the same time, in order to preserve the physical dimensions of the original theory, this procedure entails changing the dimensions of the coupling accordingly [12],

$$\sigma \rightarrow \sigma \mu^{\epsilon}. \tag{29}$$

For the particular case of the δ-function potential, its generalization (28) from D_0 to D dimensions happens to be the "obvious" one,

$$V^{(D)}(\mathbf{r}) = \sigma \mu^{\epsilon} \delta^{(D)}(\mathbf{r}). \tag{30}$$

Assuming that the two-dimensional case is considered (by selecting $D_0=2$), the regularized bound-state energy condition (26) can be directly applied, provided that $D=2-\epsilon$ and the replacement (29) is made; then,

$$\mathcal{K}_D(\mathbf{0};\kappa) = -\frac{1}{\lambda \mu^{\epsilon}}, \tag{31}$$

where $\mathcal{K}_D(\mathbf{R};\kappa)$ is defined in Eq. (14). From the small-argument behavior of the modified Bessel functions of the second kind [12,13,46]:

$$K_p(z) \overset{(z\to 0)}{\sim} \frac{1}{2}\left[\Gamma(p)\left(\frac{z}{2}\right)^{-p} + \Gamma(-p)\left(\frac{z}{2}\right)^{p}\right][1+O(z^2)] \tag{32}$$

(which amounts to a logarithmic singularity for $p=0$), the following condition is obtained [12]:

$$\frac{\lambda \mu^{\epsilon}}{4\pi}\left(\frac{2M}{\hbar^2}\frac{|E|}{4\pi}\right)^{-\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right) = 1, \tag{33}$$

which displays a simple pole at $\epsilon=0$, making the theory singular for the two-dimensional unregularized case.

Renormalization proceeds by introducing the running coupling [12,13],

$$\lambda(\epsilon) = 2\pi\epsilon\left\{1 + \frac{\epsilon}{2}[g^{(0)} - (\ln 4\pi - \gamma)]\right\}, \tag{34}$$

where γ is the Euler-Mascheroni constant; from Eq. (34), the ground-state energy becomes

$$E_{(\mathrm{gs})} = -\frac{\hbar^2\mu^2}{2M} e^{g^{(0)}}. \tag{35}$$

Notice that, as a result of the arbitrariness in the choice of $g^{(0)}$, we have the freedom to subtract, along with the pole, the term $(\ln 4\pi - \gamma)$; this is the analog of the usual modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme [51].

Finally, the residue at the pole straightforwardly provides the ground-state wave function, according to Eq. (27),

$$\Psi_{(\mathrm{gs})}(\mathbf{r}) = \frac{\kappa}{\sqrt{\pi}} K_0(\kappa r), \tag{36}$$

where

$$\kappa = \mu e^{g^{(0)}/2}, \tag{37}$$

a result in agreement with the corresponding one within the Schrödinger-equation approach [12,13]. In short, we have reproduced the familiar results: (i) the unregularized problem has a singular spectrum with a unique energy level at $-\infty$; (ii) regularization lifts this level to a finite value; (iii) renormalization provides a well-defined prescription that yields the *unique* finite ground state of the two-dimensional δ-function potential.

Most interestingly, renormalization of the two-dimensional δ-function potential leads to the emergence of an arbitrary dimensional scale, as seen in the above derivation. This remarkable phenomenon, known as dimensional transmutation [52], implies a violation of the manifest SO(2,1) symmetry of this scale-invariant potential and amounts to a simple realization of a quantum anomaly [17]. A similar symmetry analysis applies to the inverse square potential [53,54], magnetic monopole [55], and magnetic vortex [56], and has recently been generalized to the dipole potential of molecular physics [57].

V. MOMENTUM-CUTOFF REGULARIZATION

Equation (26) determines the bound-state sector of the theory, as discussed in Sec. III. An alternative regularization technique can be introduced by rewriting $G_D(\mathbf{0},\mathbf{0};E)$ in the momentum representation ($D=2$), a procedure that is equivalent to the use of the integral representations of Appendix B. In particular, the replacement of Eq. (B4) in Eq. (26) leads to the integral expression

$$\frac{\lambda}{(2\pi)^2}\int \frac{d^2q}{q^2 + (2M|E|)/\hbar^2} = 1, \tag{38}$$

where $E=-|E|$ (with $E<0$ for the possible bound states) and the variable q is a wave number.

If the integral in Eq. (38) is computed naively, an infinite result is obtained. Nevertheless, if it is generalized to D dimensions and dimensional regularization is applied, one can immediately reproduce the results of Ref. [12] and Sec. IV. Alternatively, a completely different regularization can be implemented through a momentum cutoff $\hbar\Lambda$. In fact, Λ can be introduced from the outset, directly at the level of the momentum integrals derived from the path integral and Green's function formulations. With this cutoff procedure, Eq. (38) can be straightforwardly integrated, yielding the result [15,16]

$$\frac{\lambda}{4\pi}\ln\left(\frac{\hbar^2\Lambda^2}{2M|E|}+1\right)=1; \tag{39}$$

this is equivalent to the statement

$$E_{(\mathrm{gs})}=-\frac{\hbar^2\Lambda^2}{2M}\frac{1}{e^{4\pi/\lambda}-1}\sim-\frac{\hbar^2\Lambda^2}{2M}e^{-4\pi/\lambda}. \tag{40}$$

Equations (39) and (40) lead to the same conclusions as in Sec. IV [cf. Eq. (35)], provided that $\lambda=\lambda(\Lambda)$ in such a way that $|E|$ remains finite when $\Lambda\to\infty$; this condition requires that

$$\lambda(\Lambda)=\frac{4\pi}{\ln(\Lambda^2/\kappa^2+1)}\sim-\frac{2\pi}{\ln(\kappa/\Lambda)}. \tag{41}$$

In the language of the renormalization group, this behavior leads to a Callan-Symanzik β function [51]

$$\beta(\lambda)=\Lambda\frac{d\lambda}{d\Lambda}\sim-\frac{\lambda^2}{2\pi}, \tag{42}$$

which shows the existence of an ultraviolet fixed point at zero coupling strength.

VI. REAL-SPACE REGULARIZATION

Real-space regularization may be viewed as the most "physical" regularization scheme, inasmuch as it explicitly modifies the short-distance physics in order to provide a well-defined problem. Of course, there are many possible real-space regularization schemes. Here, it proves convenient to introduce a real-space regulator a such that

$$\delta^{(2)}(\mathbf{r})\sim\frac{\delta(r-a)}{2\pi a}, \tag{43}$$

where the limit $a\to 0$ is understood. This amounts to the regularized circular δ-function potential

$$V(\mathbf{r})=-\frac{\hbar^2\lambda}{2M}\frac{\delta(r-a)}{2\pi a}\equiv g\,\delta(r-a), \tag{44}$$

which can be dealt with using the techniques developed in this paper.

Due to the central nature of Eq. (44), the formalism of Sec. II can be directly applied with the goal of obtaining the radial analogue of Eq. (25), but now with the support of the delta function at $r=a$. This can be accomplished by a procedure similar to the one used in Sec. III: (i) rewriting (19) term by term,

$$G_{l+\nu}(r'',r';E)=\sum_{n=0}^{\infty}G^{(n)}_{l+\nu}(r'',r';E); \tag{45}$$

(ii) performing the integrals at each order to obtain

$$G^{(n)}_{l+\nu}(r'',r';E)=g^n[G^{(0)}_{l+\nu}(a,a;E)]^{n-1}\times G^{(0)}_{l+\nu}(r'',a;E)G^{(0)}_{l+\nu}(a,r';E), \tag{46}$$

for $n\geqslant 1$; and (iii) summing the series, with the final result [2]

$$G_{l+\nu}(r'',r';E)=G^{(0)}_{l+\nu}(r'',r';E)-\frac{G^{(0)}_{l+\nu}(r'',a;E)G^{(0)}_{l+\nu}(a,r';E)}{G^{(0)}_{l+\nu}(a,a;E)-1/g}. \tag{47}$$

Then, the bound-state equation reads

$$\frac{\hbar^2}{2M}G^{(0)}_{l+\nu}\left(a,a;E=-\frac{\hbar^2\kappa^2}{2M}\right)=-\frac{2\pi a}{\lambda}, \tag{48}$$

which, from Eq. (20), is equivalent to

$$I_l(\kappa a)K_l(\kappa a)=\frac{2\pi}{\lambda}, \tag{49}$$

where $\nu=0$ for $D=2$.

Finally, Eq. (49) can be studied with the small-argument behavior of the modified Bessel functions of the first kind [12,13,46]

$$I_p(z)\overset{(z\to 0)}{\sim}\left(\frac{z}{2}\right)^p[\Gamma(p+1)]^{-1}[1+0(z^2)] \tag{50}$$

and of the second kind, Eq. (32). The resulting analysis is summarized by the following conclusions. If a solution were sought for $l\neq 0$, the regular boundary condition at the origin would not be satisfied, as follows from the small-argument expansion (32). Therefore, the boundary condition at the origin is *only* satisfied for s states

$$l=0. \tag{51}$$

Not surprisingly, the δ-function potential, being of zero range, can sustain bound states only in the absence of a centrifugal barrier.

For $l=0$ one then obtains the ground-state energy condition

$$E_{(\mathrm{gs})}=-\frac{\hbar^2\kappa^2}{2M}=-\frac{\hbar^2}{2M}\frac{4e^{-2\gamma}}{a^2}e^{-4\pi/\lambda}, \tag{52}$$

where again γ is the Euler-Mascheroni constant [arising from the expansion (32) for $p=0$]. Just as before, renormalization requires the running of the coupling parameter,

$$\lambda(a)=-\frac{2\pi}{\ln(\kappa a/2)+\gamma}, \tag{53}$$

an expression that reproduces the same results as in Secs. IV [Eqs. (34) and (35)] and V [Eqs. (40) and (41)]. Specifically, with the identification $a\sim 1/\Lambda$, the running couplings (41) and Eq. (53) are in exact correspondence up to finite parts, and the β function associated with Eq. (53) coincides with Eq. (42), as expected on physical grounds.

VII. SCATTERING

In this section we will consider the scattering sector of theory. First, in Sec. VII A the necessary scattering framework will be developed in D dimensions—as it relates to the theory of Sec. II—and applied to the unregularized δ-function interaction. Second, in Sec. VII B the regularization and renormalization of the two-dimensional case will be analyzed.

A. Derivation of scattering observables from infinite summation of perturbation theory

The scattering sector of theory is described most compactly by the Lippmann-Schwinger equations, which we will present in D dimensions. These equations can be directly obtained within our path-integral formulation, starting from Eq. (9), whose right-hand side stands for the real-space representation of the symbolic operator relation

$$\begin{aligned} G_D(E) &= \sum_{n=0}^{\infty} G_D^{(0)}(E)[VG_D^{(0)}(E)]^n \\ &= G_D^{(0)}(E) + G_D^{(0)}(E)VG_D(E), \end{aligned} \tag{54}$$

in which it was assumed that

$$V(\mathbf{r}'',\mathbf{r}') = \delta^{(D)}(\mathbf{r}''-\mathbf{r}')V(\mathbf{r}') \tag{55}$$

for the local interactions considered in this paper. The Lippmann-Schwinger equation (54) for the Green's function can be rewritten as a corresponding equation for the T matrix

$$\begin{aligned} T_D(E) &\equiv V + VG_D(E)V \\ &= V + VG_D^{(0)}(E)T_D(E). \end{aligned} \tag{56}$$

Moreover, Eq. (56) can be evaluated term by term with the counterpart of Eq. (23),

$$T_D(\mathbf{r}'',\mathbf{r}';E) = \sum_{n=1}^{\infty} T_D^{(n)}(\mathbf{r}'',\mathbf{r}';E), \tag{57}$$

which leads to the recursion relations

$$\begin{aligned} T_D^{(1)}(E) &= V, \\ T_D^{(n)}(E) &= VG_D^{(0)}(E)T_D^{(n-1)}(E), \end{aligned} \tag{58}$$

where $n \geqslant 2$ and the first line represents the initial condition $(n=1)$. For a δ-function interaction of the form (21), the recursion relations (58) can be applied sequentially or inductively to prove that, at order n in perturbation theory,

$$T_D^{(n)}(\mathbf{r}'',\mathbf{r}';E) = \sigma^n[G_D^{(0)}(\mathbf{0},\mathbf{0};E)]^{n-1}\delta^{(D)}(\mathbf{r}'')\delta^{(D)}(\mathbf{r}'), \tag{59}$$

which implies the bilocal form for the T matrix

$$T_D(\mathbf{r}'',\mathbf{r}';E) = \frac{1}{1/\sigma - G_D^{(0)}(\mathbf{0},\mathbf{0};E)}\delta^{(D)}(\mathbf{r}'')\delta^{(D)}(\mathbf{r}'). \tag{60}$$

Once the general framework is established, the computation of the elastic scattering amplitude $f_k^{(D)}(\Omega^{(D)})$ can be implemented in D dimensions using the familiar results of scattering theory, by studying the asymptotic behavior of the corresponding causal Green's function $\mathcal{G}_D^{(+)}(\mathbf{r}'',\mathbf{r}';E)$ [Eqs. (10) and (12)]. An alternative and more insightful albeit lengthier approach—completely based on a path-integral representation of the S matrix along the lines of Ref. [58]—will be reported elsewhere. In what follows, $\mathbf{k}'' \equiv (k'',\Omega^{(D)})$, with $\Omega^{(D)}$ being the set of hyperspherical coordinates associated with the outgoing wave vector $\mathbf{k}''$. According to the usual formulation [59] $f_k^{(D)}(\Omega^{(D)})$ is proportional to the on-shell scattering matrix elements in the momentum representation, defined as $\langle \mathbf{k}''|T_D(E)|\mathbf{k}'\rangle$, with $|\mathbf{k}'| = |\mathbf{k}''| \equiv |\mathbf{k}|$ and $E \equiv E_k = \hbar^2 k^2/2M$; explicitly,

$$\begin{aligned} f_k^{(D)}(\Omega^{(D)}) = &-\frac{1}{4\pi}\frac{2M}{\hbar^2}\left(\frac{k}{2\pi}\right)^{(D-3)/2}\int d^D r''\int d^D r' \\ &\times e^{i(\mathbf{k}'\cdot\mathbf{r}'-\mathbf{k}''\cdot\mathbf{r}'')}T_D(\mathbf{r}'',\mathbf{r}';E_k)\big|_{|\mathbf{k}'|=|\mathbf{k}''|\equiv|\mathbf{k}|}, \end{aligned} \tag{61}$$

where $\mathbf{k}'$ stands for the incident wave vector. Thus, for the contact interaction (21), substitution of the T matrix (60) in Eq. (61) yields

$$f_k^{(D)}(\Omega^{(D)}) = -\Gamma_D(k)\frac{1}{1/\lambda + \mathcal{G}_D^{(+)}(\mathbf{0};k)}, \tag{62}$$

where

$$\Gamma_D(k) = -\frac{1}{4\pi}\left(\frac{k}{2\pi}\right)^{(D-3)/2}. \tag{63}$$

Equation (62) immediately provides conclusions in agreement with the bound-state sector of the theory for an attractive potential. Let us now see how this works for $D=1$ and $D=2$.

In the one-dimensional case [with $\nu=-1/2$ and $\Omega^{(1)} \equiv \mathrm{sgn}(x)$; see comments in the paragraph after Eq. (27)], the Green's function (12) becomes the familiar function $e^{ik|x|}/2ik$ and the known transmission $T=|1+if_k(+)|^2 = (2k/\lambda)^2/[1+(2k/\lambda)^2]$ and reflection $R=|if_k(-)|^2 = 1/[1+(2k/\lambda)^2]$ coefficients are recovered [60].

In the two-dimensional case, Eq. (62) is divergent as a consequence of the small-argument limit of Eq. (12), as we will explicitly verify next.

B. Renormalization of the scattering sector

In this subsection we will apply the procedure introduced in Sec. VII A to the two-dimensional δ-function potential.

A crucial point in this approach is that, once the need for renormalization is identified, we have to show the compatibility of the renormalization procedures of both sectors (bound state and scattering). This is most easily accomplished by first renormalizing the bound-state sector and then using the corresponding running coupling to eliminate the divergences in the scattering sector. Physically, this procedure endows the theory with *predictive power*: if renormalization is implemented to control the behavior of the ground state, then the scattering observables are correspondingly and unambiguously fixed, without any free parameters.

In general, the technique of the previous subsection can be implemented for the two-dimensional δ-function interaction with each one of the regularization methods discussed in this paper. However, for the sake of simplicity, we will just derive the scattering cross section using dimensional regularization.

In the dimensional-regularization scheme, the D-dimensional expressions derived below can be viewed as the regularized counterparts of the physical $D_0=2$ case, supplemented by the renormalization of the coupling parameter. This coupling, $\lambda\mu^{\epsilon}$, can be substituted for the limit of the Green's function $\mathcal{K}_D(\mathbf{r};\kappa)$ as $r\to 0$, according to the bound-state condition (31). Then,

$$f_k^{(D)}(\Omega^{(D)})=\Gamma_D(k)\lim_{r\to 0}[\mathcal{K}_D(\mathbf{r};\kappa)-\mathcal{G}_D^{(+)}(\mathbf{r};k)]^{-1}, \quad (64)$$

where κ parametrizes the ground-state energy $|E_{(\mathrm{gs})}| =\hbar^2\kappa^2/2M$.

The limit in Eq. (64) can be obtained from the asymptotic behavior

$$\lim_{r\to 0}\mathcal{K}_D(\mathbf{r};\kappa)=-\frac{1}{(2\pi)^D}\int\frac{d^Dq}{q^2+\kappa^2}$$
$$=-\frac{1}{(4\pi)^{D/2}}\kappa^{D-2}\Gamma\left(1-\frac{D}{2}\right), \quad (65)$$

together with the analytic continuation [cf. Eq. (B6)]

$$\mathcal{G}_D^{(+)}(\mathbf{r};k)=\mathcal{K}_D(\mathbf{r};\kappa)|_{\kappa^2\to-(k^2+i\delta)}, \quad (66)$$

where $\delta=0^+$, whence

$$\lim_{r\to 0}\mathcal{G}_D^{(+)}(\mathbf{r};k)=\frac{1}{(2\pi)^D}\int\frac{d^Dq}{k^2-q^2+i\delta}$$
$$=-\frac{1}{(4\pi)^{D/2}}(-k^2-i\delta)^{D/2-1}\Gamma\left(1-\frac{D}{2}\right). \quad (67)$$

Finally, this implies that

$$\lim_{r\to 0}[\mathcal{G}_D^{(+)}(\mathbf{r};k)-\mathcal{K}_D(\mathbf{r};\kappa)]$$
$$=\frac{1}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)[\kappa^{D-2}-(-k^2-i\delta)^{D/2-1}], \quad (68)$$

whose limit $\epsilon\to 0^+$, with $D=2-\epsilon$, becomes

$$\lim_{\epsilon\to 0}\lim_{r\to 0}[\mathcal{G}_D^{(+)}(\mathbf{r};k)-\mathcal{K}_D(\mathbf{r};\kappa)]=-\frac{1}{4\pi}(\ln\kappa^2-\ln k^2+i\pi), \quad (69)$$

because of the identity $\ln[-(k^2+i\delta)]=\ln k^2-i\pi$ for the principal branch of the natural logarithm. Equation (69) finally provides the scattering amplitude (64) as a function of the incident energy $E_k=\hbar^2k^2/2M$,

$$f_k^{(2)}(\Omega^{(2)})=\sqrt{\frac{2\pi}{k}}\left[\ln\left(\frac{E_k}{|E_{(\mathrm{gs})}|}\right)-i\pi\right]^{-1}, \quad (70)$$

where, upon taking the limit $\epsilon\to 0$, the observable (70) is evaluated in dimension $D_0=2$.

Equation (70) is parametrized with the ground-state energy $E_{(\mathrm{gs})}$, which is a dimensional variable that has replaced the original dimensionless coupling—the phenomenon of dimensional transmutation [52]. In addition, Eq. (70) verifies the expected *isotropic* scattering of a contact interaction: the two-dimensional δ-function interaction scatters only s waves, with a scattering phase shift $(l=0)$

$$\tan\delta_0^{(2)}(k)=\frac{\pi}{\ln(E_k/|E_{(\mathrm{gs})}|)}, \quad (71)$$

and with phase shifts $\delta_l^{(2)}(k)=0$ for all $l\neq 0$. These results can be summarized with the diagonal S matrix in the angular momentum representation,

$$S_{l,l'}^{(2)}(E)=\delta_{l0}\delta_{l'0}\frac{\ln(E/|E_{(\mathrm{gs})}|)+i\pi}{\ln(E/|E_{(\mathrm{gs})}|)-i\pi}. \quad (72)$$

As a final step, by direct integration of the differential scattering cross section $d\sigma^{(2)}(E_k,\Omega^{(2)})/d\Omega_2=|f_k^{(2)}(\Omega^{(2)})|^2$, one obtains the total scattering cross section

$$\sigma_2(E)=\frac{4\pi^2}{k}\frac{1}{[\ln(E/|E_{(\mathrm{gs})}|)]^2+\pi^2}. \quad (73)$$

Remarkably, all the scattering observables display a logarithmic dependence with respect to the incident energy, and with a characteristic scale set by the ground state. Finally, two relevant checks can be made.

(i) The unique pole of the scattering matrix (72) provides the ground-state energy.

(ii) Levinson's theorem,

$$\delta_l^{(2)}(k=0)-\delta_l^{(2)}(k=\infty)=\pi\mathcal{N}_l, \quad (74)$$

provides the correct number of bound states: $\mathcal{N}_0=1$ and $\mathcal{N}_l=0$ for $l>0$.

VIII. CONCLUSIONS

In summary, we have completed a thorough analysis of the path-integral derivation for the two-dimensional δ-function interaction, including renormalization *à la* field theory and the compatibility of the bound-state and scattering sectors. Our results are in agreement with the previously known ones from the Schrödinger-equation approach. The formalism provided in this paper also allows for generalizations to arbitrary dimensionalities and further study of the three-dimensional case—which is the nonrelativistic limit of the scalar ϕ^4 field theory and is relevant for the question of triviality [29].

Our path-integral treatment of the contact interaction confirms the following conclusions.

(i) The problem of singular potentials and bound states is best dealt with by means of the energy Green's function $G(E)$.

(ii) Infinite summations and resummations of perturbation theory give the required nonperturbative behavior.

(iii) Proper analytic continuations may be needed in certain regimes.

(iv) Renormalization can be implemented in the bound-state sector to uniquely predict the scattering observables.

(v) The effective-field-theory program, which leads to singular potentials, requires renormalization in a quantum-mechanical setting, such as the one presented in this paper.

Extensions of this generic program to other singular potentials and field theory will be presented elsewhere.

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APPENDIX A: FREE-PARTICLE PROPAGATOR AND GREEN'S FUNCTION WITHIN THE PATH-INTEGRAL FORMULATION

For a free particle, the integrals involved in Eq. (2) can be directly performed. The result is exact even before taking the large-N limit, as it is independent of the order N of the partition of the time lattice,

$$K_D^{(0)}(\mathbf{r}'',\mathbf{r}';T)=\left(\frac{M}{2\pi i T\hbar}\right)^{D/2}\exp\left[\frac{iM(\mathbf{r}''-\mathbf{r}')^2}{2\hbar T}\right]. \quad \text{(A1)}$$

The corresponding energy Green's function can be computed from Eq. (4), so that

$$G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E)=\frac{1}{i\hbar}\left(\frac{M}{2\pi i\hbar}\right)^{D/2}\mathcal{I}_\nu(\tau(R,E),\omega(E)), \quad \text{(A2)}$$

where [61]

$$\mathcal{I}_\nu(\tau,\omega)=\int_0^\infty dT\,T^{-\nu-1}\exp\left\{i\omega\left[T+\frac{\tau^2}{T}\right]\right\}$$
$$=i\pi\tau^{-\nu}\exp\left(i\frac{\nu\pi}{2}\right)H_\nu^{(1)}(2\omega\tau), \quad \text{(A3)}$$

in which ν is given by Eq. (13), $\omega(E)=E/\hbar$, and $\tau(R,E)$ is defined from $\omega(E)[\tau(R,E)]^2=MR^2/2\hbar$, with $\mathbf{R}=\mathbf{r}-\mathbf{r}'$. As a result, rewriting $2\omega\tau=kR$, the Green's function becomes

$$G_D^{(0)}(\mathbf{r}'',\mathbf{r}';E)=-\frac{i}{4}\left(\frac{2M}{\hbar^2}\right)\left(\frac{k}{2\pi R}\right)^{\nu}H_\nu^{(1)}(kR), \quad \text{(A4)}$$

which is equivalent to Eqs. (10) and (12). Finally, the analytic continuations (14) and (66) of the Green's function to negative energies—needed for the bound-state sector—can be straightforwardly obtained with the relation [46]

$$K_\nu(\mp iz)=\pm\frac{\pi i}{2}e^{\pm i\pi\nu/2}H_\nu^{(1,2)}(z). \quad \text{(A5)}$$

The corresponding expressions for the free particle in hyperspherical coordinates can be established from Eq. (16), which gives

$$K_{l+\nu}(r'',r';T)=\lim_{N\to\infty}\sqrt{r'r''}\,\alpha^N\prod_{k=1}^{N-1}\left[\int_0^\infty dr_k r_k\right]$$
$$\times e^{-\alpha(r_1^2+\cdots r_{N-1}^2)}I_{l+\nu}(\alpha r_0 r_1)\cdots$$
$$\times I_{l+\nu}(\alpha r_{N-1}r_N), \quad \text{(A6)}$$

where $\alpha=M/i\hbar\epsilon$. Equation (A6) can be evaluated recursively by repeated application of Weber's second exponential formula [45], with the result [9]

$$K_{l+\nu}(r'',r';T)=\frac{M}{i\hbar T}\sqrt{r'r''}\exp\left[\frac{iM}{2\hbar T}(r'^2+r''^2)\right]$$
$$\times I_{l+\nu}\left(\frac{Mr'r''}{i\hbar T}\right), \quad \text{(A7)}$$

whence the Green's function becomes (with the symbol $a=M/2\hbar$),

$$G_{l+\nu}(r'',r';T)=-\frac{2a}{\hbar}\sqrt{r'r''}\int_0^\infty\frac{dT}{T}I_{l+\nu}\left(\frac{2ar''r'}{iT}\right)$$
$$\times\exp\left\{i\left[\frac{(E+i0^+)}{\hbar}T+a(r'^2+r''^2)\frac{1}{T}\right]\right\}, \quad \text{(A8)}$$

which can integrated in closed form with the substitution $u=1/T$ [62] to yield Eq. (20).

APPENDIX B: FREE-PARTICLE GREEN'S FUNCTIONS FROM OPERATOR FORMULATION

The connection between the path-integral and operator formulations of the Green's function can be established by means of Eq. (6). In particular, the latter can be recast into the form of a Green-Helmholtz equation, which we will consider next. For the free-particle case, which applies to unbounded space, translational invariance implies that $G_D(\mathbf{r},\mathbf{r}';E_k)$ is only a function of $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, and the rescaling (10) can be defined. Then, the Green-Helmholtz equation reads

$$[\nabla^2_{\mathbf{R},D}+k^2]\mathcal{G}_D(\mathbf{R};k)=\delta^{(D)}(\mathbf{R}). \tag{B1}$$

As is well known, its Fourier transform $\tilde{\mathcal{G}}_D(\mathbf{q};k)=(k^2-q^2)^{-1}$ leads to an ill-defined integral expression that needs to be evaluated by an additional prescription defining the boundary conditions at infinity; for outgoing $(+)$ and incoming $(-)$ boundary conditions,

$$\mathcal{G}^{(\pm)}_D(\mathbf{R};k)=\int\frac{d^Dq}{(2\pi)^D}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{k^2-q^2\pm i\delta}$$
$$=(2\pi)^{-D/2}R^{-(D/2-1)}\int_0^\infty\frac{q^{D/2}J_{D/2-1}(qR)}{k^2-q^2\pm i\delta}dq, \tag{B2}$$

where $\delta=0^+$. The energy dependence of Eq. (6) shows that $\mathcal{G}^{(+)}_D(\mathbf{R};k)$ is the function that reproduces the correct behavior of the path-integral expression (5).

Similarly, analytic continuation to negative energies leads to the modified Green-Helmholtz equation,

$$[\nabla^2_{\mathbf{R},D}-\kappa^2]\mathcal{K}_D(\mathbf{R};\kappa)=\delta^{(D)}(\mathbf{R}), \tag{B3}$$

where the analog of the rescaling (10) should be considered. The Fourier transform of $\mathcal{K}_D(\mathbf{R};\kappa)$ can be derived from Eq. (B3), i.e., $\tilde{\mathcal{K}}_D(\mathbf{q};\kappa)=-(q^2+\kappa^2)^{-1}$, a result that can be finally inverted to give

$$\mathcal{K}_D(\mathbf{R};\kappa)=-\int\frac{d^Dq}{(2\pi)^D}\frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{q^2+\kappa^2}$$
$$=-(2\pi)^{-D/2}R^{-(D/2-1)}\int_0^\infty\frac{q^{D/2}J_{D/2-1}(qR)}{q^2+\kappa^2}dq. \tag{B4}$$

Equation (B4) can be evaluated in terms of the modified Bessel function of the second kind $K_\nu(\kappa R)$, of order $\nu=D/2-1$, i.e., [63],

$$\mathcal{K}_D(\mathbf{R};\kappa)=-\frac{1}{2\pi}\left(\frac{\kappa}{2\pi R}\right)^\nu K_\nu(\kappa R). \tag{B5}$$

Furthermore, Eq. (B2) can be expressed in terms of Hankel functions of order ν; for example, Eq. (B1) can be obtained from Eq. (B3) with the replacement $\kappa=\mp ik$, which provides the solution

$$\mathcal{G}^{(\pm)}_D(\mathbf{R};k)=\mathcal{K}_D(\mathbf{R};\kappa=\mp ik), \tag{B6}$$

and the choice of signs amounts to the choice of boundary conditions at infinity or the $i\delta$ prescription. From the identity (A5), it follows that Eq. (B6) acquires the form

$$\mathcal{G}^{(\pm)}_D(\mathbf{R};k)=\mp\frac{i}{4}\left(\frac{k}{2\pi R}\right)^\nu H^{(1,2)}_\nu(kR), \tag{B7}$$

a result that reduces to familiar expressions for $D=1,2,3$.

[1] W.M. Frank, D.J. Land, and R.M. Spector, Rev. Mod. Phys. **43**, 36 (1971), and references therein.
[2] H.E. Camblong and C.R. Ordóñez, e-print hep-th/0109003.
[3] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1995), Vol. I, p. 560, and references therein.
[4] A recent approach based on the Feynman-Schwinger representation is introduced in Ç. Şavkli, J. Tjon, and F. Gross, Phys. Rev. C **60**, 055210 (1999); **61**, 069901(E) (2000).
[5] M. Consoli and P.M. Stevenson, e-print hep-ph/9711449; Int. J. Mod. Phys. A **15**, 133 (2000). In these references, a $1/r^3$ potential is shown to arise from radiative corrections.
[6] S.R. Beane *et al.*, Phys. Rev. A **64**, 042103 (2001).
[7] G.P. Lepage, e-print nucl-th/9706029; U. van Kolck, Prog. Part. Nucl. Phys. **43**, 337 (1999); S.R. Beane, P.F. Bedaque, M.J. Savage, and U. van Kolck, Nucl. Phys. A **700**, 377 (2002).
[8] K.V. Bhagwat and S.V. Lawande, Phys. Lett. A **141**, 321 (1989).
[9] C. Grosche and F. Steiner, *Handbook of Feynman Path Integrals* (Springer-Verlag, Berlin, 1998), and references therein.
[10] K.S. Gupta and S.G. Rajeev, Phys. Rev. D **48**, 5940 (1993).
[11] H.E. Camblong, L.N. Epele, H. Fanchiotti, and C.A. García Canal, Phys. Rev. Lett. **85**, 1590 (2000).
[12] H.E. Camblong, L.N. Epele, H. Fanchiotti, and C.A. García Canal, Ann. Phys. (N.Y.) **287**, 14 (2001).
[13] H.E. Camblong, L.N. Epele, H. Fanchiotti, and C.A. García Canal, Ann. Phys. (N.Y.) **287**, 57 (2001).
[14] The relevance of this problem has been highlighted in the following classic references. N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions*, 2nd ed. (Oxford University Press, Oxford, 1949), p. 40; K.M. Case, Phys. Rev. **80**, 797 (1950); L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, Oxford, 1977), p. 114; P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 2, p. 1665; H. A. Bethe and R. Jackiw, *Intermediate Quantum Mechanics*, 3rd ed. (Addison-Wesley, Reading, MA, 1986), pp. 263–265.
[15] C. Thorn, Phys. Rev. D **19**, 639 (1979).
[16] K. Huang, *Quarks, Leptons, and Gauge Fields*, 2nd ed. (World Scientific, Singapore, 1992), Secs. 10.7 and 10.8.
[17] R. Jackiw, in *M. A. B. Bég Memorial Volume*, edited by A. Ali and P. Hoodbhoy (World Scientific, Singapore, 1991).
[18] H. Bethe and R. Peierls, Proc. R. Soc. London, Ser. A **148**, 146 (1935); E. Fermi, Ric Sci. **7**, 13 (1936); English translation, E. Fermi, *Collected Papers*, *Vol. 1*, *Italy 1921–1938* (University of Chicago Press, Chicago, 1962).
[19] G. Breit, Phys. Rev. **71**, 215 (1947); G. Breit and P.R. Zilsel, *ibid.* **71**, 232 (1947); J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952).

[20] For recent applications, see A.T. Kruppa, K. Varga, and J. Révai, Phys. Rev. C **63**, 064301 (2001), and references therein.

[21] G.F. Koster and J.C. Slater, Phys. Rev. **96**, 1208 (1954); J. Friedel, Nuovo Cimento, Suppl. **7**, 287 (1958); A.M. Clogston, Phys. Rev. **125**, 439 (1962); S. Doniach and E. H. Sondheimer, *Green's Functions for Solid State Physicists* (Benjamin, Reading, MA, 1974).

[22] E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).

[23] Yu. Demkov and V. N. Ostrovskii, *Zero-Range Potentials and Their Applications in Atomic Physics* (Plenum, New York, 1989).

[24] Ya.B. Zel'dovich, J. Exp. Theor. Phys. (U.S.S.R.) **38**, 819 (1960) [Sov. Phys. JETP **11**, 594 (1960)]; F.A. Berezin and L.D. Faddeev, Sov. Math. Dokl. **2**, 372 (1961).

[25] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (Springer-Verlag, New York, 1988).

[26] K. Wódkiewicz, Phys. Rev. A **43**, 68 (1991).

[27] A.S. Goldhaber, Phys. Rev. D **16**, 1815 (1977); C.J. Callias, *ibid.* **16**, 3068 (1977); M. Carreau, E. Farhi, and S. Gutmann, *ibid.* **42**, 1194 (1990); D.K. Park, J. Math. Phys. **36**, 5453 (1995); J. Phys. A **29**, 6407 (1996); D.K. Park and S.-K. Yoo, e-print hep-th/9712134; J.M. Roman and R. Tarrach, J. Phys. A **29**, 6073 (1996); E.R. Karat and M.B. Schulz, Ann. Phys. (N.Y.) **254**, 11 (1997).

[28] P. deSousa Gerbert, Phys. Rev. D **40**, 1346 (1989).

[29] M.A.B. Bég and R.C. Furlong, Phys. Rev. D **31**, 1370 (1985); K. Huang, Int. J. Mod. Phys. A **4**, 1037 (1989).

[30] D.R. Phillips, S.R. Beane, and T.D. Cohen, Ann. Phys. (N.Y.) **263**, 255 (1998).

[31] I. Mitra, A. DasGupta, and B. Dutta-Roy, Am. J. Phys. **66**, 1101 (1998).

[32] R.J. Henderson and S.G. Rajeev, J. Math. Phys. **39**, 749 (1998).

[33] S.-L. Nyeo, Am. J. Phys. **68**, 571 (2000).

[34] S.K. Adhikari and T. Frederico, Phys. Rev. Lett. **74**, 4572 (1995).

[35] C. Manuel and R. Tarrach, Phys. Lett. B **328**, 113 (1994).

[36] P. Gosdzinsky and R. Tarrach, Am. J. Phys. **59**, 70 (1991).

[37] L.R. Mead and J. Godines, Am. J. Phys. **59**, 935 (1991).

[38] J.F. Perez and F.A.B. Coutinho, Am. J. Phys. **59**, 52 (1991).

[39] B. Holstein, Am. J. Phys. **61**, 142 (1993).

[40] A. Cabo, J.L. Lucio, and H. Mercado, Am. J. Phys. **66**, 240 (1998).

[41] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 2nd ed. (World Scientific, Singapore, 1995), and references therein.

[42] C. Grosche, Phys. Rev. Lett. **71**, 1 (1993).

[43] In the time-lattice version of expression (8), at order n in perturbation theory, one introduces $n+1$ subintervals T_α—with each subinterval partitioned in the usual way into N_α parts. Notice the extra integrations with respect to time in Eq. (8) [compared with Eq. (9)], which arise when separating the end points of each subinterval.

[44] *Higher Transcendental Functions*, edited by A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi (McGraw-Hill, New York, 1953), Vol. 2, Chap. XI.

[45] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. (Cambridge University Press, Cambridge, England, 1944).

[46] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover Publications, New York, 1972).

[47] Equation (16) explicitly exhibits interdimensional dependence, i.e., D and l appear in the combination $l+\nu$; e.g., J. H. Van Vleck, in *Wave Mechanics, the First Fifty Years*, edited by W. C. Price *et al.* (Butterworth, London, 1973), p. 26.

[48] R. W. Robinett, *Quantum Mechanics: Classical Results, Modern Systems, and Visualized Examples* (Oxford University Press, Oxford, 1997), Secs. 9.1.2 and 9.1.3.

[49] C.G. Bollini and J.J. Giambiagi, Nuovo Cimento Soc. Ital. Fis., **12B**, 20 (1972); G. 't Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972); J.F. Ashmore, Lett. Nuovo Cimento **4**, 289 (1972).

[50] G. Leibbrandt, Rev. Mod. Phys. **47**, 849 (1975).

[51] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).

[52] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).

[53] R. Jackiw, Phys. Today **25**(1), 23 (1972).

[54] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento Soc. Ital. Fis., **34A**, 569 (1976).

[55] R. Jackiw, Ann. Phys. (N.Y.) **129**, 183 (1980).

[56] R. Jackiw, Ann. Phys. (N.Y.) **201**, 83 (1990).

[57] H.E. Camblong, L.N. Epele, H. Fanchiotti, and C.A. García Phys. Rev. Lett. **87**, 220402 (2001).

[58] C.C. Gerry and V.A. Singh, Phys. Rev. D **21**, 2979 (1980).

[59] J. R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions* (Wiley, New York, 1972).

[60] S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, New York, 1974).

[61] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. (Academic Press, New York, 2000), p. 363, formula 3.471.11.

[62] Ref. [61], p. 700, formula 6.635.3.

[63] Ref. [61], p. 671, formula 6.566.2.