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Semiclassical methods in curved spacetime and black hole thermodynamics

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Improved semiclassical techniques are developed and applied to a treatment of a real scalar field in a D-dimensional gravitational background. This analysis, leading to a derivation of the thermodynamics of black holes, is based on the simultaneous use of (i) a near-horizon description of the scalar field in terms of conformal quantum mechanics; (ii) a novel generalized WKB framework; and (iii) curved-spacetime phase-space methods. In addition, this improved semiclassical approach is shown to be asymptotically exact in the presence of hierarchical expansions of a near-horizon type. Most importantly, this analysis further supports the claim that the thermodynamics of black holes is induced by their near-horizon conformal invariance.

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I. INTRODUCTION

The fundamental concepts of black hole thermodynamics have been confirmed within several frameworks since the 1970s [1], including in string theory [2] and loop quantum gravity [3]. In particular, the Bekenstein-Hawking entropy $S_{\rm BH}$ [4] and the Hawking effect [5] suggest that the horizon plays a fundamental role in black hole thermodynamics [6,7], an idea that has been emphasized in recent approaches [1,8] and generalized to the holographic principle [9,10]. The connections between the horizon quantum features and the thermodynamics include the existence of a near-horizon conformal symmetry [11–16]. In Ref. [17] we have discussed the emergence of this thermodynamic behavior, within a *semiclassical approximation* with the following building blocks: (i) the near-horizon conformal symmetry; (ii) the competition of the field angular-momentum degrees of freedom; and (iii) the singular dynamics of conformal quantum mechanics [18,19]. Given the *singularity of the conformal potential*, these ingredients suggest the questions:

(i) Is the use of semiclassical techniques justified within conformal quantum mechanics [20]?
(ii) Is there a preferred order for the angular-momentum expansion *vis-à-vis* the radial conformal analysis?

In this paper, we give an affirmative answer to the first question through an improved semiclassical method, and show that the stage at which the field angular-momentum expansion is introduced is immaterial; thus, our framework justifies the use of these concepts.

In Sec. II we survey the scalar field equations, including their near-horizon properties. In Sec. III we develop a generalized version of the semiclassical WKB method in the presence of a hierarchical expansion—the near-horizon expansion being a particular case. In Sec. IV we use phase-space methods in curved spacetime to derive the spectral function needed for the thermodynamics. Finally, in Sec. V, we discuss and critically reexamine this framework.

II. FIELD EQUATIONS

We will consider a real scalar field Φ, with mass m, in a D-dimensional spacetime ($D \geq 4$), defined through its action (with the metric conventions of Ref. [21])

$$S = -\frac{1}{2}\int d^D x\sqrt{-g}[g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + m^2\Phi^2 + \xi R\Phi^2], \tag{1}$$

where ξ is its coupling to the curvature scalar R, and the static spacetime metric is

$$\begin{aligned} ds^2 &= g_{00}(\vec{x})dt^2 + \gamma_{ij}(\vec{x})dx^i dx^j \\ &= -f(r)dt^2 + [f(r)]^{-1}dr^2 + r^2 d\Omega^2_{(D-2)}, \end{aligned} \tag{2}$$

where $d\Omega^2_{(D-2)}$ is the metric on S^{D-2}. The derivation of the thermodynamics requires counting the field modes for the spectral number function $N(\omega)$ leading to the entropy; thus, this procedure is based on the combinatorics of the modes of the Euler-Lagrange equation

$$\Box\Phi - (m^2 + \xi R)\Phi = 0 \tag{3}$$

satisfied by Φ from the action (1). The quantum field operator can be expanded as

$$\Phi(t, \vec{x}) = \sum_s [a_s\phi_s(\vec{x})e^{-i\omega_s t} + a_s^\dagger\phi_s^*(\vec{x})e^{i\omega_s t}], \tag{4}$$

where $a_s^\dagger$ and a_s are the creation and annihilation operators, and $\phi_s(\vec{x})$ is a complete set of orthonormal functions; the separation of the time coordinate of Eq. (4) turns Eq. (3) into

$$\check{\Delta}_{(\gamma)}\phi + \gamma^{ij}\partial_i(\ln\sqrt{|g_{00}|})\partial_j\phi + \Im_{(0)}(r;\omega)\phi = 0, \tag{5}$$

where $\check{\Delta}_{(\gamma)}$ is the Laplace-Beltrami operator with respect to the spatial metric $\gamma_{ij}(\vec{x})$ and

$$\Im_{(0)}(r;\omega) = \frac{\omega^2}{f(r)} - (m^2 + \xi R). \tag{6}$$

An important ingredient for our thermodynamic analysis is the *near-horizon expansion*, which is defined with respect to the coordinate

$$x = r - r_+, \tag{7}$$

where $r = r_+$ selects the event horizon $\mathcal{H}$ from the largest root of the scale-factor equation $f(r) = 0$ (excluding cosmological horizons). Given a quantity Q, for a leading-order s, it will prove useful to use the notation $Q(r) \overset{(\mathcal{H})}{\sim} Q_+^{(s)} x^s/\Gamma(s+1)$, which amounts to performing a Laurent expansion; in the case of a Taylor series expansion, $Q_+^{(s)}$ stands for the sth order derivative of $Q(r)$ at $\mathcal{H}$. In particular, we will consider the parameter

$$f'_+ \equiv f'(r_+), \tag{8}$$

with $f'_+ \neq 0$ for the *nonextremal* case; this entails the leading-order expansion $f(r) \overset{(\mathcal{H})}{\sim} f'_+ x[1 + O(x)]$, and its corresponding higher orders $f'(r) \overset{(\mathcal{H})}{\sim} f'_+$ and $f''(r) \overset{(\mathcal{H})}{\sim} f''_+$.

Several paths can be taken to describe the relevant physics of the modes. The first one involves the Liouville transformation [22] $\phi(\vec{x}) = |g_{00}|^{-1/4}\Psi(\vec{x})$, so that Eq. (5) becomes

$$\check{\Delta}_{(\gamma)}\Psi + \Im(r;\omega)\Psi = 0, \tag{9}$$

whose normal form involves the extra terms

$$\Im_{(1)}(r) = f(r)\left\{\frac{1}{16}\left[\frac{f'(r)}{f(r)}\right]^2 - \frac{D-2}{4r}\frac{f'(r)}{f(r)} - \frac{1}{4}\frac{f''(r)}{f(r)}\right\}, \tag{10}$$

in addition to the original function $\Im_{(0)}(r;\omega)$ of Eq. (6), with

$$\Im(r;\omega) = \Im_{(0)}(r;\omega) + \Im_{(1)}(r). \tag{11}$$

Finally, the near-horizon expansion of Eq. (9) involves the conformally-symmetric terms

$$\Im(r;\omega) \overset{(\mathcal{H})}{\sim} f'_+\left(\Theta^2 + \frac{1}{16}\right)\frac{1}{x}[1 + O(x)], \tag{12}$$

with the same scaling as the Laplace-Beltrami operator and characterized by the parameter

$$\Theta = \frac{\omega}{f'_+}. \tag{13}$$

The second path consists of introducing the spherical symmetry of the metric (2) directly from the outset, so that Eq. (5) turns into

$$f(r)\phi'' + \left[f'(r) + (D-2)\frac{f(r)}{r}\right]\phi' - \frac{1}{r^2}\hat{\ell}^2\phi + \Im_{(0)}(r;\omega)\phi = 0, \tag{14}$$

where $-\hat{\ell}^2 = -\hat{\ell}^a\hat{\ell}_a$ stands for the Laplacian on S^{D-2}, with its spherical-harmonic eigenfunctions $Y_{lm}(\Omega)$. In addition, applying the Liouville transformation [22]

$$\phi_s(\vec{x}) \equiv \phi_{nlm}(r,\Omega) = Y_{lm}(\Omega)\chi(r)u_{nl}(r) \tag{15}$$

[where $s = (n, l, m)$], with $\chi(r) = [f(r)]^{-1/2}r^{-(D-2)/2}$, the radial equation becomes

$$u''(r) + \Im(r;\omega,\alpha_{l,D})u(r) = 0, \tag{16}$$

in which

$$\Im(r;\omega,\alpha_{l,D}) = I(r;\omega) - \frac{1}{f(r)}\frac{l(l+D-3)}{r^2} \tag{17}$$

$$= \frac{\Im_{(0)}(r;\omega)}{f(r)} + \left\{\left[\frac{1}{f(r)} - 1\right]\nu^2 + \frac{1}{4}\right\}\frac{1}{r^2} + R_{rr} + \frac{1}{4}\left[\frac{f'(r)}{f(r)}\right]^2 - \frac{1}{f(r)}\frac{\alpha_{l,D}}{r^2} \tag{18}$$

includes the terms $I(r;\omega)$ associated with the radial Liouville transformation, while

$$\alpha_{l,D} = l(l+D-3) + \nu^2 = \left(l + \frac{D-3}{2}\right)^2 \tag{19}$$

is the angular-momentum coupling. In Eqs. (18) and (19), $\nu = (D-3)/2$ and

$$R_{rr} = -\frac{1}{2}\frac{f''(r)}{f(r)} - \frac{(D-2)}{2r}\frac{f'(r)}{f(r)} \tag{20}$$

is the radial component of the Ricci tensor for the metric (2).

The most important property of Eq. (18) is that its near-horizon expansion,

$$I(r;\omega) \overset{(\mathcal{H})}{\sim} \left(\Theta^2 + \frac{1}{4}\right)\frac{1}{x^2}[1 + O(x)], \tag{21}$$

is *conformal* because $I(r;\omega)$ has the same scale dimension as the second-order derivative in Eq. (16). Comparison of Eqs. (12) and (21) shows that: (i) the scale dimension is changed from $1/x$ to $1/x^2$; and (ii) the numerical term has changed from 1/16 to 1/4. The first point is due to a rearrangement of factors: Eqs. (12) and (21) describe the same physics within different *coordinate representations* of conformal quantum mechanics. The second, subtler point is crucial for the counting of modes, as will be seen in Secs. III and IV.

Finally, including the angular momentum, the near-horizon expansion of Eq. (17) is

$$\Im(r;\omega,\alpha_{l,D}) \stackrel{(\mathcal{H})}{\sim} \left\{\left[\frac{\omega^2}{(f'_+)^2}+\frac{1}{4}\right]\frac{1}{x^2}-\frac{\alpha_{l,D}}{f'_+ r_+^2}\frac{1}{x}\right\} \times [1+O(x)], \tag{22}$$

which displays the properties: (i) the leading term is the strong-coupling potential

$$V_{\rm eff}(x) \stackrel{(\mathcal{H})}{\sim} -\left(\Theta^2+\frac{1}{4}\right)\frac{1}{x^2}[1+O(x)] \tag{23}$$

of conformal quantum mechanics [17]; (ii) the angular-momentum term is still required for the correct statistical counting of modes leading to the thermodynamics [17].

III. NEAR-HORIZON GENERALIZED WKB FRAMEWORK

We will consider the effective problem obtained after separation of the time coordinate, which consists of a d-dimensional equation (with spacetime dimensionality $D = d + 1$)

$$\check{\Delta}_{(\gamma)}\Psi + \Im(\vec{x})\Psi = 0. \tag{24}$$

A. Covariant WKB method: General formulation

Our goal is to select a WKB wave vector that would reproduce the original Eq. (24) as closely as possible. Without loss of generality, one can start from a WKB-type solution

$$\Psi_{\rm WKB}(\vec{x}) = A(\vec{x})\exp\left[i\int^{\vec{x}} k_j(\vec{x}')dx'^j\right], \tag{25}$$

in which the wave number $k_j(\vec{x})$ and amplitude $A(\vec{x})$ are real. This is known to be a first-order approximation in an expansion with respect to the "small" parameter $\hbar$ but may fail to be an exact solution of the problem (24). However, defining

$$\tilde{\Im}(\vec{x}) = \| \vec{k}(\vec{x}) \|^2 \equiv \gamma^{jh}(\vec{x})k_j(\vec{x})k_h(\vec{x}), \tag{26}$$

the wave function (25) satisfies the *exact* equation

$$\check{\Delta}_{(\gamma)}\Psi_{\rm WKB} + [\tilde{\Im}(\vec{x}) - Q(\vec{x})]\Psi_{\rm WKB} = 0, \tag{27}$$

which follows by enforcing the conservation of the "effective probability current" $j_h = A^2 k_h$:

$$\nabla_j[\gamma^{jh}A^2k_h] \equiv \frac{1}{\sqrt{\gamma}}\partial_j[\gamma^{jh}\sqrt{\gamma}A^2k_h] = 0, \tag{28}$$

thus suppressing the terms associated with imaginary coefficients, and leading to

$$Q(\vec{x}) = \frac{\check{\Delta}_{(\gamma)}A(\vec{x})}{A(\vec{x})}. \tag{29}$$

Traditionally, the function $Q(\vec{x})$ in Eq. (29) is viewed as the "error" in approximating $\Psi(\vec{x})$ with $\Psi_{\rm WKB}(\vec{x})$, with applicability limited by $|Q(\vec{x})| \ll \| \vec{k}(\vec{x}) \|^2$. However, for the near behavior $x \sim 0$, a modified WKB approach, in the style first proposed by Langer [23], may be needed. We will consider a generalized covariant scheme that expands the range of applications and permits a treatment of the coordinate singularity. In this proposal, the additional term $Q(\vec{x})$ in Eq. (27) is absorbed by the original function $\Im(\vec{x})$ in Eq. (24), in such a way that $\Psi_{\rm WKB}(\vec{x}) = \Psi(\vec{x})$; thus, from Eqs. (24) and (27) it follows that

$$\tilde{\Im}(\vec{x}) = \Im(\vec{x}) + Q[\tilde{\Im}](\vec{x}), \tag{30}$$

which is an auxiliary equation, where $Q(\vec{x})$ depends on the unknown $\tilde{\Im}(\vec{x})$ and its derivatives. Thus, the improved WKB method amounts to the replacement $\Im(\vec{x}) \to \tilde{\Im}(\vec{x})$, where the subtraction of the "quantum potential" $Q(\vec{x})$ generates an *effective potential* $-\tilde{\Im}(\vec{x})$ that captures the relevant physics. In this viewpoint, Eqs. (26) and (28)–(30) constitute a set of coupled partial differential equations; even though an exact solution to this combined system is not generally available, a systematic approximation scheme can be developed as follows. Specifically, Eq. (30) is taken as the starting point of a successive-approximation scheme

$$\tilde{\Im}^{(n)}(\vec{x}) = \Im(\vec{x}) + Q[\tilde{\Im}^{(n-1)}](\vec{x}), \tag{31}$$

which begins at zeroth order ($n = 0$) with the standard WKB approximation

$$\tilde{\Im}^{(0)}(\vec{x}) = \Im(\vec{x}). \tag{32}$$

We now turn to the development of a novel approximation framework, which follows when this scheme is applied concurrently with an expansion of the near-horizon type.

B. Generalized WKB framework in the presence of a hierarchical expansion

As discussed throughout this paper, the emergence of black hole thermodynamics is governed by the near-horizon behavior of the metric (2), which can be displayed by means of an expansion with respect to the coordinate x of Eq. (7). The existence of an expansion of this kind furnishes a hierarchy, which organizes the relevant physics with respect to powers of the variable x, starting with the dominant physics for the leading order. Such a *hierarchical expansion* can be conveniently applied concurrently with the (covariant) WKB approach of the previous subsection to provide a systematic modified WKB approach. As we will show next, within the ensuing *hierarchical WKB framework*, the first-order approximation ($n = 1$) in Eq. (31) becomes *asymptotically exact* with respect to $x \sim 0$, so that

$$\tilde{\Im}(\vec{x}) \sim \tilde{\Im}^{(1)}(\vec{x}) = \Im(\vec{x}) + Q[\Im](\vec{x}), \tag{33}$$

where $\sim$ stands for the hierarchical expansion [with the near-horizon case being $\stackrel{(\mathcal{H})}{\sim}$].

The dominant physics is described by the leading orders of the building blocks of Eq. (24): $\Im(x)$ and $\check{\Delta}_{(\gamma)}$. In the hierarchical WKB framework, the relevant expansion variable is x, which we choose with dimensions of length. Then, the *leading scale dimensions* of $\Im(x)$, $\check{\Delta}_{(\gamma)}$, and other variables can be identified from the homogeneous degree of the leading-order terms, under a rescaling $x \to \lambda x$. Specifically, the dimension $[\check{\Delta}_{(\gamma)}] = -p$ can be extracted from

$$\frac{\check{\Delta}_{(\gamma)} F(x)}{F(x)} \sim \chi(s) x^p, \tag{34}$$

while $[\Im(x)] = -q$ is defined by

$$\Im(x) \sim c x^q. \tag{35}$$

In Eq. (34), the test function $F(x)$ admits the expansion $F(x) \sim F^{(s)} x^s / \Gamma(s+1)$, while $\chi(s)$ is a normalization factor that depends on the dimension parameter s associated with $F(x)$. The scale dimension of $Q(x)$ can be determined from Eqs. (29) and (34) (with $A \equiv F$),

$$Q(x) \sim \chi(s) x^p, \tag{36}$$

where the normalization function $\chi(s)$ is to be computed from the specific expansion of the operator $\check{\Delta}_{(\gamma)}$ with respect to x; thus, the "quantum potential" $Q(x)$ has the same scale dimension, $-p$, as the Laplace-Beltrami operator. As a result, Eq. (30) defines the scale dimension of $\tilde{\Im}(x)$ by selecting the leading order, i.e., $[\tilde{\Im}(x)] = -\min\{p, q\}$.

In addition to the scale dimension displayed in Eq. (36), it is necessary to determine the normalization prefactor $\chi(s)$, whose functional form can be computed from the derivatives in Eq. (29). However, the actual value of s requires the use of the continuity condition (28), combined with Eq. (26) (see the near-horizon expansion in the next subsection).

The nature of the expansion leads to three possible scenarios from a comparison of the scale dimensions of $\Im(x)$ and $\check{\Delta}_{(\gamma)}$: *regular case*, defined by $q > p$, so that the Laplace-Beltrami operator yields the dominant physics near $x \sim 0$; *properly singular case*, defined by $q < p$, so that $\Im(x)$ is dominant as $x \sim 0$; *marginally singular case*, defined by $q = p$, so that $\Im(x)$ and the Laplace-Beltrami operator [along with the quantum potential $Q(x)$] compete at the same order. As for the solutions, for the regular case, they are of power-law free-particle type as $x \sim 0$; in addition, for the singular cases $q \leq p$, asymptotically exact WKB solutions can be found by:

(1) The *standard WKB method*, for the properly singular case. In this method, the required effective potential $-\tilde{\Im}(\vec{x})$ only involves the term $-\Im(\vec{x})$, with negligible $Q(\vec{x})$.

(2) The *improved WKB method*, which applies to the marginally singular case. In this method, the required effective potential $-\tilde{\Im}(\vec{x})$ is given from the rule (30) or (33).

The latter, nontrivial case can be established by going back to Eq. (31) and verifying it becomes self-consistent at the $n = 1$ level, in the form of Eq. (33). Moreover, substituting Eqs. (35) and (36) in Eq. (33), and defining $c^{(*)} = -\chi(s)$, we see that

$$\tilde{\Im}(x) \sim [c - c^{(*)}] x^p. \tag{37}$$

Thus, the nature of the modes changes around $c = c^{(*)}$, which plays the role of a critical coupling, with c selecting either a singular (supercritical) or regular (subcritical) behavior.

In conclusion, singular quantum mechanics can be described with asymptotic exactness by the improved WKB method, with modes having a semiclassical appearance due to the singular term $\Im(x)$. However, in the marginally singular case, the competing "potential" $Q(x)$ generates the subtraction of a critical coupling, as in Eq. (37), and the leading physics has *asymptotic scale invariance*—this applies to nonextremal metrics in the near-horizon expansion.

C. Near-horizon WKB framework: Multidimensional case

The multidimensional Eq. (24) describes the full-fledged spatial dependence of the modes. The near-horizon expansion of the Laplace-Beltrami operator for the metric (2),

$$\check{\Delta}_{(\gamma)} = \frac{1}{\sqrt{\gamma}} \partial_j [\sqrt{\gamma} \gamma^{jk} \partial_k] \stackrel{(\mathcal{H})}{\sim} f'_+ \left(x \partial_x^2 + \frac{1}{2} \partial_x \right), \tag{38}$$

implies that $p = -1$ (or "$\check{\Delta}_{(\gamma)} \propto x^{-1}$"); therefore, if $A(x) \propto x^s$, then

$$Q(x) = \frac{\check{\Delta}_{(\gamma)} A(x)}{A(x)} \stackrel{(\mathcal{H})}{\sim} f'_+ s \left(s - \frac{1}{2} \right) x^{-1}. \tag{39}$$

Clearly, for the nonextremal metrics, the leading scale dimension of $\Im(x) \stackrel{(\mathcal{H})}{\propto} 1/x$ is equal to 1, thus showing that this is a marginally singular case: the near-horizon physics exhibits *SO(2,1) conformal invariance*. Accordingly, the semiclassical function $\tilde{\Im}(x)$ is given by Eq. (37) with a critical coupling

$$c^{(*)} = f'_+ s \left(\frac{1}{2} - s \right). \tag{40}$$

In addition, the value of the parameter s for the multidimensional case can be determined from the continuity Eq. (28), which yields the leading order of the amplitude through

$$\frac{\partial}{\partial x}\left(\frac{\gamma^{xx}}{\sqrt{f}}A^2k_x\right)\overset{(\mathcal{H})}{\propto}\frac{\partial}{\partial x}(\sqrt{x}A^2k_x)\overset{(\mathcal{H})}{\sim}0. \tag{41}$$

Therefore, $\hat{k}_x \equiv \sqrt{\gamma^{xx}}k_x = \sqrt{f}k_x \overset{(\mathcal{H})}{\propto} \sqrt{x}\,k_x$ gives the amplitude scaling

$$A(x)\overset{(\mathcal{H})}{\propto}(\hat{k}_x)^{-1/2}\overset{(\mathcal{H})}{\propto}[\tilde{\Im}(x)]^{-1/4}\overset{(\mathcal{H})}{\propto}x^{1/4}, \tag{42}$$

where, from Eq. (26), $\tilde{\Im}(x)\overset{(\mathcal{H})}{\sim}\gamma^{xx}(k_x)^2 = (\hat{k}_x)^2\overset{(\mathcal{H})}{\propto}1/x$ ($p = q = -1$). In particular, Eq. (42) implies that $s = 1/4$; as a result, from Eq. (39), the leading "extra term" becomes

$$Q(x)\overset{(\mathcal{H})}{\sim} - f'_+\frac{1}{16}\frac{1}{x}\overset{(\mathcal{H})}{\sim} - f(x)\frac{1}{16}\frac{1}{x^2}. \tag{43}$$

Finally, from the near-horizon expansion of Eqs. (12), (33), and (43),

$$\tilde{k} \equiv \sqrt{\frac{\tilde{\Im}(x)}{f(x)}}\overset{(\mathcal{H})}{\sim}\sqrt{\frac{\Im(x) + Q[\Im](x)}{f(x)}}\overset{(\mathcal{H})}{\sim}\frac{\Theta}{x}, \tag{44}$$

which defines an improved wave number. Then, the leading form of $\tilde{\Im}(x)$ yields the chain of relations, $\tilde{k}\overset{(\mathcal{H})}{\sim}k_r\overset{(\mathcal{H})}{\sim}k_{\rm conf}(x)$, which reduce to the *conformal wave number*

$$k_{\rm conf}(x) \equiv \frac{\Theta}{x}. \tag{45}$$

In conclusion, this calculation shows that: (i) the leading covariant momentum component is radial; (ii) $k_{\rm conf}(x)$ embodies the improved WKB features of conformal quantum mechanics; and (iii) $k_{\rm conf}(x)$ is the correct input for the phase-space algorithms of Sec. IV.

D. Near-horizon WKB framework: Reduced radial case

Equation (16) was derived through the sequence of exact Liouville transformations; in turn, this equation can be solved within the semiclassical approximation, with

$$\Delta_{(1D)} \equiv \partial_x^2 \tag{46}$$

applied to the formalism of Sec. III B. The original radial part of the Laplacian also includes the prefactor $f(r)$; however, in the sequence of transformations leading to Eq. (16), $f(r)$ was scaled away. As a result, the leading scaling of Eq. (46) is now given by

$$Q(x) = \frac{A''(x)}{A(x)} = s(s-1)x^{-2}, \tag{47}$$

i.e., "$\Delta_{(1D)} \propto x^{-2}$." Moreover, the near-horizon leading form of Eq. (16) becomes

$$\Delta_{(1D)}u(x) + \left[\frac{\Theta^2 + 1/4}{x^2}\right]u(x) = 0, \tag{48}$$

which corresponds to the effective conformal interaction (23) and implies that

$$\Im(x)\overset{(\mathcal{H})}{\sim}\left(\Theta^2 + \frac{1}{4}\right)\frac{1}{x^2}. \tag{49}$$

Accordingly, in the radial setup of the generalized WKB framework, the scale dimensions of Eqs. (46), (47), and (49) are equal to 2 for the nonextremal metrics: the near-horizon physics is marginally singular, with the scale symmetry of conformal quantum mechanics.

In addition, the parameter s is determined from the leading-order continuity equation,

$$\frac{\partial}{\partial x}(A^2k_x)\overset{(\mathcal{H})}{\sim}\frac{\partial}{\partial x}\{A^2[\tilde{\Im}(x)]^{1/2}\}\overset{(\mathcal{H})}{\sim}0, \tag{50}$$

where $k_x\overset{(\mathcal{H})}{\sim}[\tilde{\Im}(x)]^{1/2}$ for the one-dimensional analogue of Eq. (26). In turn,

$$A(x)\overset{(\mathcal{H})}{\propto}[\tilde{\Im}(x)]^{-1/4}\overset{(\mathcal{H})}{\propto}x^{1/2}, \tag{51}$$

because $\tilde{\Im}(x) \propto 1/x^2$ for the reduced radial case (as $p = q = -2$). In particular, Eq. (51) shows that $s = 1/2$ and yields the critical coupling $c^{(*)} = 1/4$, as Eq. (47) turns into

$$Q(x)\overset{(\mathcal{H})}{\sim} - \frac{1}{4x^2}. \tag{52}$$

Thus, the one-dimensional analogue of Eqs. (26) and (33) yields the conformal behavior (45),

$$k_{\alpha_{l,D}}(r) \equiv \sqrt{\tilde{\Im}(r;\omega,\alpha_{l,D})}\overset{(\mathcal{H})}{\sim}k_{\rm conf}(x). \tag{53}$$

In conclusion, Eq. (53) provides the wave number for the WKB wave functions

$$u_{\pm}(r) = [k_{\alpha_{l,D}}(r)]^{-1/2}\exp\left[\pm i\int^r k_{\alpha_{l,D}}(r')dr'\right]. \tag{54}$$

Even though the variables $\tilde{k}$ of Eq. (44) and $k_{\alpha_{l,D}}(r)$ of Eq. (53) are different, their near-horizon leading contributions reduce to the same conformal value (45). Moreover, Eq. (22) implies the competition of the angular momenta with $k_{\rm conf}(x)$ in the form

$$k_{\alpha_{l,D}}(r = r_+ + x;\Theta,\alpha_{l,D})\overset{(\mathcal{H})}{\sim}k_{\rm conf}(x)\sqrt{1 - \frac{\alpha_{l,D}x}{f'_+r_+^2\Theta^2}}. \tag{55}$$

IV. PHASE-SPACE METHODS FOR QUANTUM MECHANICS AND QUANTUM FIELD THEORY IN CURVED SPACETIME

The main goal of this section is to derive phase-space expressions—compatible with the improved WKB approach—for the cumulative number of modes or spectral function

$$N(\omega) = \sum_{\substack{s \\ \omega_s \le \omega}} 1. \quad (56)$$

For a monotonic increasing operator [24] $-\hat{\mathcal{H}}_{\text{eff}}(\omega)$, Eq. (56) is equivalent to

$$N(\omega) = \text{Tr}[\theta(-\hat{\mathcal{H}}_{\text{eff}}(\omega))] = \sum_s \theta(-[\hat{\mathcal{H}}_{\text{eff}}(\omega)]_s), \quad (57)$$

in which $\theta(z)$ stands for the Heaviside function and the formal trace is defined in the Hilbert space spanned by the basis of modes $\phi_s(\vec{x})$.

A. Phase-space method: Generic techniques

For the statistical mechanics of a quantum-mechanical system in *curved space*, the semiclassical counterpart of Eq. (57) is derived by counting the number of phase-space cells $d\Gamma/(2\pi)^d$ enclosed within a given ω-parametrized surface $\mathcal{H}_{\text{eff}}(\vec{x}, \vec{p}; \omega) = 0$; this is computed with the Liouville measure in local Darboux coordinates [25] $d\Gamma = dx^1 \wedge \ldots \wedge dx^d \wedge dp_1 \wedge \ldots \wedge dp_d$—with the shorthand $d\Gamma = d^d x d^d p$, in terms of the covariant momentum components. Then, for a classical Hamiltonian $\mathcal{H}_{\text{eff}}(\vec{x}, \vec{p})$, with configuration-space metric $\gamma_{ij}(\vec{x})$,

$$N(\omega) \approx \int \frac{d\Gamma}{(2\pi)^d} \theta(-\mathcal{H}_{\text{eff}}(\vec{x}, \vec{p}; \omega)) = \frac{1}{(2\pi)^d} \int d^d x \sqrt{\gamma} \int_{\mathcal{H}_{\text{eff}}(\vec{x}, \vec{p}; \omega) \le 0} d^d p \frac{1}{\sqrt{\gamma}} \quad (58)$$

(where the symbol $\approx$ denotes the semiclassical approximation *before a hierarchical expansion*).

For the analysis of the black hole problem of Eq. (9), the momentum dependence of the effective Hamiltonian $\hat{\mathcal{H}}_{\text{eff}}(\omega)$ is merely quadratic and two distinct ways of evaluating Eq. (58) are possible: (i) the *multidimensional* approach and (ii) the *reduced radial* approach.

The multidimensional approach starts by integrating out *all* the generalized momenta:

$$N(\omega) \approx \frac{\Omega_{(d-1)}}{d(2\pi)^d} \int dV_{(d)} \parallel \vec{k}(\vec{x}) \parallel^d, \quad (59)$$

where $dV_{(d)} = d^d x \sqrt{\gamma}$ is the d-dimensional spatial volume element and $\parallel \vec{k}(\vec{x}) \parallel \equiv \parallel \vec{p}(\vec{x}) \parallel = \sqrt{\gamma^{jh}(\vec{x}) p_j(\vec{x}) p_h(\vec{x})}$ (with $\vec{k} \equiv \vec{p}$). In addition, in the presence of a hierarchical expansion [from Eqs. (26) and (33)] $\parallel \vec{k}(\vec{x}) \parallel \sim \{\Im(\vec{x}) + Q[\Im](\vec{x})\}^{1/2}$, with $\approx$ replaced by $\sim$. Moreover, when the potential is spherically symmetric: $\tilde{\Im}(\vec{x}) = \tilde{\Im}(r)$, Eq. (59) becomes

$$N(\omega) \approx \frac{[\Omega_{(d-1)}]^2}{d(2\pi)^d} \int dr [\gamma_{rr}]^{-(d-1)/2} r^{d-1} [\tilde{k}(r)]^d, \quad (60)$$

where

$$\tilde{k}(r) = [\gamma_{rr}]^{1/2} \parallel \vec{k}(r) \parallel. \quad (61)$$

In the radial approach, for a spherically symmetric Hamiltonian, Eq. (58) turns into a radial integral in configuration space and an integral over the angular momenta; this is accomplished by a four-step method. First, from the polar coordinates $\vec{x} \equiv (r, \theta^1, \cdots, \theta^{d-1})$, the conjugate momenta $\vec{p} \equiv (p_r, \ell_1, \cdots, \ell_{d-1})$ satisfy $\gamma^{jk} p_j p_k = \gamma^{rr} p_r^2 + \ell^2/r^2$, where ℓ_a are angular momenta (with $a = 1, \cdots, d-1$) and $\ell^2 = \ell^a \ell_a$. Second, the radial momentum can be integrated out with $\int_{-\infty}^{\infty} dp_r \theta(\tilde{\Im}(r) - \gamma^{rr} p_r^2 - \alpha_l/r^2) = 2\tilde{k}(r; \alpha_l)$, where $\alpha_l = \ell^2$ and

$$\tilde{k}(r; \alpha_l) \equiv k_{\alpha_l, D}(r) = (\gamma^{rr})^{-1/2} \sqrt{\tilde{\Im}(\vec{x}) - \frac{\alpha_l}{r^2}} = \sqrt{[\tilde{k}(r)]^2 - \gamma_{rr} \frac{\alpha_l}{r^2}}, \quad (62)$$

with $\tilde{k}(r)$ defined by Eq. (61). Third, the angular-momentum dependence is kept through $\alpha \equiv \alpha_l$ and with the use of $\int d^{d-1}\ell/\sqrt{\sigma} = \Omega_{(d-2)} \int d\alpha \alpha^{(d-3)/2}/2$, where σ_{ab} is the S^{d-1} metric associated with $\Omega \equiv \{\theta^a\}$ ($a = 1, \ldots, d-1$). Finally, integration of the angular variables $d\Omega_{(d-1)} \equiv d^{d-1}\theta\sqrt{\sigma}$ yields the solid angle $\Omega_{(d-1)}$. Thus, the spectral function becomes

$$N(\omega) \approx \frac{\Omega_{(d-1)}\Omega_{(d-2)}}{(2\pi)^d} \int d\alpha \alpha^{(d-3)/2} \int_I dr \tilde{k}(r; \alpha), \quad (63)$$

where the interval I is bounded by the classical turning points; in the presence of a hierarchical expansion, Eq. (63) requires the use of *improved wave numbers*. Equivalently, Eq. (63) has been shown to follow from the one-dimensional Sturm oscillation theorems [17].

B. Phase-space method: Quantum field theory in curved spacetime and near-horizon physics

We now turn to the specific computation of the spectral number function $N(\omega)$ corresponding to our quantum field theory in *curved spacetime*. The starting point is the spatially reduced Klein-Gordon Eq. (9). Its classical limit involves a simple Hamiltonian formulation with the modification (30) at the level of the effective potential. Consequently,

$$N(\omega) \approx \int d^d x \int \frac{d^d p}{(2\pi)^d} \theta(\Im_{(0)}(r; \omega) + \Im_{(1)}(r) + Q(r) - \gamma^{jk}(\vec{x}) p_j p_k), \quad (64)$$

where the "quantum potential" $Q(r)$ is required for the near-horizon expansion of nonextremal metrics, and the approaches of the previous subsection can be applied.

In the multidimensional approach, Eq. (64) leads to the counterpart of Eqs. (59) and (60) with $\gamma_{rr}(r) = 1/f(r)$; in the near-horizon limit, from Eq. (44),

$$N(\omega)\overset{(\mathcal{H})}{\sim}\frac{1}{d2^{d-2}[\Gamma(d/2)]^2}\times\int dx r_+^{d-1}\underbrace{[f'_+x]^{(d-1)/2}}_{\text{angular contribution}}\underbrace{[k_{\rm conf}(x)]^d}_{\text{conformal interaction}}, \quad (65)$$

which displays a competition of the conformal wave number $k_{\rm conf}(x)$ with the angular-momentum factors $[f'_+x]^{(d-1)/2}$. These factors reduce the degree of divergence of the integral, but the ensuing singular behavior can be ultimately attributed to the ultraviolet singularity of conformal quantum mechanics [20]. As a final step, from Eqs. (45) and (65),

$$N(\omega)\overset{(\mathcal{H})}{\sim}\frac{1}{\pi\Gamma(d-1)}B\left(\frac{d-1}{2},\frac{3}{2}\right)\Theta^d[f'_+r_+^2]^{(d-1)/2}\times\lim_{a\to 0}\int_a^{x_1}\frac{dx}{x^{(d+1)/2}}, \quad (66)$$

where a is a radial cutoff and $B(p,q)$ is the beta function, while x_1 is an arbitrary upper limit (with a scale of the order of r_+). This cutoff and the associated renormalization of Eq. (66) are discussed in the next section and analyzed in Ref. [17].

In a similar manner, for the reduced radial problem, Eq. (55) turns Eq. (63) into

$$N(\omega)\overset{(\mathcal{H})}{\sim}\frac{1}{\pi\Gamma(d-1)}\int_0^{\alpha_{\max}}d\alpha\,\alpha^{d/2-3/2}\int_I dx k_{\rm conf}(x)\times\sqrt{1-\frac{\alpha x}{f'_+r_+^2\Theta^2}}, \quad (67)$$

where $\alpha_{\max}=\alpha_{\max}(a)=\Theta^2f'_+r_+^2/a$ is the angular-momentum cutoff arising from the passage of the right turning point through $r=a$. Finally, reversing the order of integration and using a beta-function identity, Eq. (66) follows again. This shows the equivalence of the reduced radial and multidimensional approaches.

V. BEKENSTEIN-HAWKING ENTROPY FROM THE NEAR-HORIZON EXPANSION: CONCLUSIONS

In this paper we have illustrated the use of improved semiclassical techniques for the computation of spectral functions and derived the corresponding near-horizon expansions, with the central result (66) being independent of the semiclassical procedure involved.

Unfortunately, as it stands, Eq. (66) appears to be divergent when the lower limit a approaches zero. This singularity can be traced to the scale invariance of the effective conformal interaction and is inherited by the thermodynamic observables. The cutoff a serves as a regulator and leads to the renormalization of the theory, which can be implemented geometrically by absorbing the coordinate assignment a into a distance or "elevation"

$$h_D=\int_{r_+}^{r_++a}ds\overset{(\mathcal{H})}{\sim}\frac{2\sqrt{a}}{\sqrt{f'_+}} \quad (68)$$

from the horizon. As shown in Ref. [17], the various contributions to the entropy can be organized into those factors that are purely conformal and those arising from the angular momentum: their interplay leads to the familiar Bekenstein-Hawking entropy $S_{\rm BH}=\mathcal{A}/4$, which is a $(D-2)$-dimensional "hypersurface" feature, induced by the horizon. Furthermore, this result relies on the purely conformal characterization of the Hawking temperature [17,26] needed in the statistical-mechanical calculations. Moreover, this procedure shows that the "new physics" of a full-fledged quantum gravitational theory arises from within an invariant distance of the order of the Planck length.

Despite its appealing features discussed above, the regularization procedure based on the brick-wall model leaves a number of unanswered questions. First, the computation leading to the Bekenstein-Hawking result, with the correct numerical prefactor of $1/4$, involves a fine tuning of the cutoff [1,6,7]. This poses a problem: in this method of calculation, the numerical prefactor appears to depend upon the number Z of species of particles, rather than being a Z-independent value of 1/4; for example, in the case of Z scalar fields, the required fine tuning involved in Eq. (68) would lead to a brick-wall elevation with the species dependence

$$h_D=\frac{1}{2}[D\zeta(D)\Gamma(D/2-1)\pi^{1-3D/2}Z]^{1/(D-2)}. \quad (69)$$

Another paradoxical feature of the entropy computed by a brick-wall method is that it can be absorbed by a renormalization of Newton's gravitational constant G_N, as shown in Refs. [27,28]. However, in this light, it is possible that the species problem associated with the entropy prefactor may be compensated by a corresponding Z-dependent renormalization of Newton's constant [7,29,30].

In summary, we have established that the procedure that singles out the leading conformal behavior also provides a systematic application of *semiclassical methods*. The robust nature of this framework and the asymptotically exact semiclassical description of conformal quantum mechanics are somewhat surprising, given the presence of a coordinate singularity. Remarkably, these techniques: (i) converge towards a *unique* result driven by the near-horizon expansion, the Bekenstein-Hawking entropy; (ii) point to the horizon degrees of freedom that determine the thermodynamics. The fact that this universality is driven by the near-horizon symmetry is intriguing, but its deeper geometrical meaning is not well understood. However, the robustness and simplicity of these properties suggest their possible origin from an even more fundamental principle of nature.

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