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Degree Spectra of the Successor Relation of Computable Linear Orderings

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Abstract. We establish that for every computably enumerable (c.e.) Turing degree $\mathbf{b}$, the upper cone of c.e. Turing degrees determined by $\mathbf{b}$ is the degree spectrum of the successor relation of some computable linear ordering. This follows from our main result, that for a large class of linear orderings, the degree spectrum of the successor relation is closed upward in the c.e. Turing degrees.

1. Introduction and Preliminaries

The effective properties of countable structures and relations on these structures have been thoroughly studied in recent decades. Of course, it is most interesting to consider natural structures and relations. Here, we focus on the successor relation of computable linear orderings. A linear ordering L is *computable* if its universe, $|L|$, is computable and L has a computable ordering relation. If L is infinite, we may assume that its domain is the set $\mathbb{N}$ of natural numbers. In general, a structure with domain $\mathbb{N}$ is computable if its atomic diagram is computable.

Our terminology and notation for computability theoretic notions are as in Soare [12] and Odifreddi [8], and those particular to linear orderings and computable structures are as in Rosenstein [9] and Ash-Knight [1]. We write ω for the usual order type of $\mathbb{N}$, and η for the order type of the rational numbers $\mathbb{Q}$. At times we abuse notation and write $L \cong \omega$ to indicate that the order type of the linear ordering L is ω. For a linear ordering L, L^* denotes the reverse ordering.

We write $\deg(A)$ for the Turing degree of the set A, and $\mathcal{R}$ for the set of all computably enumerable (c.e.) Turing degrees. For a c.e. degree $\mathbf{a}$, the *upper cone of c.e. degrees* determined by $\mathbf{a}$ is

$$\mathcal{R}(\geq \mathbf{a}) = \{\mathbf{b} \in \mathcal{R} \mid \mathbf{a} \leq \mathbf{b}\}.$$

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For a linear ordering L, the *successor relation* of L, $Succ_L$, is defined as follows: for $a, b \in |L|$,

$$Succ_L(a,b) \iff a <_L b \ \wedge \neg\exists c(a <_L c <_L b).$$

An element (a, b) of the successor relation is called a *successor pair*. We consider this relation in the context of the following definition.

Definition 1.1 (Harizanov [7]). Let S be a relation on the domain of a computable structure M. The (Turing) *degree spectrum* of S on M is the set

$$DgSp_M(S) = \{\deg(f(S)) \mid f : M \cong M', \text{ and } M' \text{ is computable}\}.$$

For a computability theoretic class $\mathcal{C}$, we say that the relation S is *intrinsically* $\mathcal{C}$ *on* M if the image of S under any isomorphism from M to another computable structure belongs to $\mathcal{C}$. The successor relation of a computable linear ordering is intrinsically co-c.e., so its degree spectrum must always be contained in the c.e. degrees.

There are two known examples of singleton degree spectra of the successor relation. One is trivial. Namely, if L has only finitely many successor pairs, then

$$DgSp_L(Succ_L) = \{\mathbf{0}\},$$

in other words, $Succ_L$ is intrinsically computable. In fact, in this case L is computably categorical ([5], [11]), that is, for every computable copy M of L, there is a computable isomorphism from L to M. Downey and Moses [4] constructed the other known singleton example: a linear ordering L having a successor relation with degree spectrum

$$DgSp_L(Succ_L) = \{\mathbf{0}'\},$$

so here $Succ_L$ is intrinsically complete. This example is an immediate consequence of the following theorem.

Theorem 1.2 (Downey and Moses [4]). *For any non-computable c.e. set C, there is a computable linear ordering L such that $Succ_L \equiv_T C$ and $C \leq_T Succ_{L'}$ for every computable linear ordering $L' \cong L$. Furthermore, L has the form*

$$L = I_0 + L_0 + I_1 + L_1 + \dots,$$

where each I_i is a block of length $i + 3$ and L_i has order type η or $(\eta + 2 + \eta) \cdot \tau$ for some τ.

The other extreme, where the degree spectrum of the successor relation contains all c.e. degrees, is realized in the following example. This result follows from a general theorem in [6], but we give an easy direct proof here for the reader's convenience.

Example 1.3. For any c.e. set A, there is a linear ordering $L \cong \omega$ so that $Succ_L \equiv_T A$. In other words,

$$DgSp_\omega(Succ_\omega) = \mathcal{R}(\geq 0).$$

Proof. Let A be an infinite c.e. set, and suppose that $\{A_s\}_{s\in\mathbb{N}}$ is a computable sequence of finite sets such that $A_s \subseteq A_{s+1}$, $A = \cup_s A_s$, $A_0 = \emptyset$, and $|A_{s+1} - A_s| = 1$. Let $0 <_L 2 <_L 4 <_L \dots$, and declare $2n <_L 2s+1 <_L 2n+2$ if $n \in A_{s+1} - A_s$.

It is easy to see that $(\mathbb{N}, <_L)$ is a computable linear ordering, $(\mathbb{N}, <_L) \cong \omega$, and $Succ_L \equiv_T A$. □

2. Main Result

We establish that for a large class of computable linear orderings, the degree spectrum of the successor relation is closed upward in the c.e. degrees.

Theorem 2.1. *Let L be a computable linear ordering with domain $\mathbb{N}$ such that the following condition holds:*

(U) *for every $x \in \mathbb{N}$ there are $a, b \in \mathbb{N}$ with $Succ_L(a,b)$ and $x <_L a$.*

Let A be a c.e. set so that $Succ_L \leq_T A$. Then there exists a computable linear ordering $M \cong L$ with $Succ_M \equiv_T A$.

Proof. Let L be a computable linear ordering satisfying condition (U), and $L_0 \subset L_1 \subset \cdots$ be a computable approximation of L such that each L_{i+1} is finite and has at least one element $<_L$-greater than all elements of L_i. Assume A is a c.e. set with $Succ_L \leq_T A$, and that it is non-computable. Let $a_0, a_1, \dots$ be a $1-1$ computable enumeration of A.

We build a computable $M \cong L$ such that $Succ_M \equiv_T A$. This M will be constructed by finite approximation $(M_s)_{s\in\omega}$, with $M_0 \subset M_1 \subset \cdots$ and $M = \bigcup_s M_s$. Natural numbers are added to M in numerical order, so the universe of M is $\mathbb{N}$, and is hence computable.

At each stage s of the construction we specify the linear ordering $<_M$ on $|M_s|$, which will determine an isomorphism $f_s : M_s \to L_{n_s}$, for some n_s. Hence, for $m, m' \in |M_s|$, $m <_M m' \iff f_s(m) <_L f_s(m')$.

For notational convenience, let $l_0^s, \dots, l_{k_s}^s$ be the elements of the set L_{n_s} in increasing $<_L$ order, and $m_0^s, \dots, m_{k_s}^s$ be the elements of M_s in increasing $<_M$ order. It will also be convenient to take $l_{-1}^s <_L x <_L l_0^s$ and $l_{k_s}^s <_L x <_L l_{k_s+1}^s$ to simply mean $x <_L l_0^s$ and $l_{k_s}^s <_L x$, respectively. We adopt a similar convention for the elements of M_s. Thus for all $j \leq k_s$, $f_s(l_j^s) = m_j^s$.

Define $r(s) = m^s_{k_s}$ (the $<_M$-largest element of M_s) for each s. This strictly increasing computable function will play the role of a restraint in the construction.

Construction[1]

Stage 0. Let $n_0 = 0$, $M_0 = L_0 = L_{n_0}$, and f_0 be the identity on M_0.

Stage $s + 1$. From the previous stage, we have $f_s : M_s \cong L_{n_s}$.

Case 1. If $a_s \geq s$, define $n_{s+1} = n_s+1$ and add new elements to M_s to obtain an M_{s+1} for which there is an isomorphism $f_{s+1} : M_{s+1} \cong L_{n_{s+1}}$ extending f_s:

> Let $\{z_0 < z_1 < \ldots < z_j\}$ be the elements of $L_{n_{s+1}} - L_{n_s}$ in the usual order, and $k_{s+1} = card(L_{n_{s+1}}) + 1$. Let m be the least natural number not in $|M_s|$. For each $i \leq j$, if $l_k <_L z_i <_L l_{k+1}$ for some $-1 \leq k \leq k_{s+1}$, then declare $m_k <_M m + i < m_{k+1}$. Set $f_{s+1} = f_s \cup \{(m + i, z_i)\}_{i \leq j}$.

Case 2. When $a_s < s$, we have a two-step action. First we extend M_s by breaking existing successor pairs beyond the restraint $r(a_s)$, and then extend to an appropriate isomorphism. For every successor pair (x, y) of M_s such that $r(a_s) \leq_{M_s} x <_{M_s} y$, insert a new natural number $<_M$-between x and y to obtain $M'_s \supseteq M_s$:

> Let m be the least natural number not in $|M_s|$, and let t be such that $m^s_t = r(a_s)$, and $k = card(|M_s|) - 1$. For each $0 \leq i < k - t$, declare $m^s_{t+i} <_{M'_s} m + i <_{M'_s} m^s_{t+i+1}$.

Next, find the least $n_{s+1} > n_s$ such that there is an embedding $f'_s : M'_s \to L_{n_{s+1}}$ with $f'_s(x) = f_s(x)$ for all $x \leq_{M_s} r(a_s)$ in $|M_s|$, and $f'_s(x) \geq_L f_s(x)$ for all other $x \in |M_s|$. Such an n_{s+1} exists because L has no rightmost element. We can then add new elements to M'_s to obtain M_{s+1} for which there is an isomorphism $f_{s+1} : M_{s+1} \cong L_{n_{s+1}}$ extending f'_s via the same process used in Case 1 above.

This completes the construction.

Note that since $M_0 \subset M_1 \subset M_2 \subset \cdots$ is a computable sequence of finite linear orderings, $M = \cup_s M_s$ is a computable linear ordering. Also, since $n_{s+1} > n_s$, $\lim_s n_s = +\infty$ and $L = \cup_s L_{n_s}$.

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[1]This is a modification of our original construction, and we are grateful to the referee for suggesting simplifications.

We proceed to show that $f = \lim_s f_s$ exists and is an isomorphism from M to L. At the same time, we show that f is computable from A. Subsequently, we establish $A \equiv_T Succ_M$.

In demonstrating these facts, we make use of a function h defined as follows. Given x, let s be the least such that $x \in |M_s|$. Define $h(x) \geq s$ to be the least such that $A \upharpoonright s+1 = A_{h(x)} \upharpoonright s+1$. Note that $x \in M_{h(x)}$, and h is an A-computable function.

Lemma 2.2. *The function $f = \lim_s f_s$ exists, f is an isomorphism, and $f \leq_T A$.*

Proof. Given x, choose s so that $x \in M_s$ and note that $x \leq_M r(s)$. Then for each $t \geq h(r(s))$, we have $f_{t+1}(x) = f_t(x)$ (in fact, this is true of all $y \in M_t$ such that $y \leq_M r(s)$). Thus, the limit $f(x) = \lim_s f_s(x)$ exists. Furthermore, at each stage s, f_s is order-preserving, and as a result f is as well.

To see that f is an isomorphism, it remains to establish bijectivity. Observe that, by construction, we have for all s that $f_s(x) \leq_L f_{s+1}(x) \leq_L f(x)$. Thus, for any $y \in L_{n_s}$, we have $y \leq_L f_s(r(s)) \leq_L f_{h(r(s))}(r(s)) = f_t(r(s)) = f(r(s))$ for all $t \geq h(r(s))$. Hence, $f_{t+1}^{-1}(y) = f_t^{-1}(y)$ for any such y and all $t \geq h(r(s))$. Because $\lim_s n_s = +\infty$, f must be a bijection.

Since h is an A-computable function and r is computable, it is clear that f is A-computable. □

Lemma 2.3. $A \equiv_T Succ_M$.

Proof. We have $(x, y) \in Succ_M$ if and only if $(f(x), f(y)) \in Succ_L$. Since A computes both f and $Succ_L$, it follows that $Succ_M \leq_T A$.

Conversely, to determine whether $n \in A$, let s be such that for some $(x, y) \in Succ_M$, we have $r(n) \leq_{M_s} x <_{M_s} y$, with both x and y in M_s. Since $M \cong L$ by Lemma 2.2, M satisfies condition (U) as well and such an s exists. Note that since at least one element is added to M at each stage, $n < s$. If $n = a_t$ for some later stage t, $s \leq t$, then the construction ensures that $(x, y) \notin Succ_M$. Consequently, $n \in A$ if and only if $n = a_t$ for some $t < s$, and hence $A \leq_T Succ_M$. □

This completes the proof of Theorem 2.1.

□

The result in Theorem 2.1 applies to a somewhat broader class of linear orderings than just those satisfying condition (U). First, for any linear ordering L, the degree spectrum of the successor relation in L will be identical to that of L^*, so a descending sequence of successor pairs satisfying a symmetric condition (U^*) is similarly sufficient.

Additionally, suppose that L is a computable linear ordering in which (U) does not hold. Then L may be decomposed as $L = L_2 + L_1$, where L_1 has order type 1 or $1 + \eta$. The ordering L_2 is an initial segment of L and is computable since it is definable with a quantifier-free formula, and its successor relation is at most finitely different from that of L. Consequently, $DgSp_L(Succ_L) = DgSp_{L_2}(Succ_{L_2})$, and if L_2 satisfies (U), $DgSp_L(Succ_L)$ will be closed upward in the c.e. degrees.

This process may be iterated any finite number of times to obtain a computable initial segment L' of L with $DgSp_L(Succ_L) = DgSp_{L'}(Succ_{L'})$. If L' satisfies (U), then $DgSp_{L'}(Succ_{L'})$, and hence $DgSp_L(Succ_L)$, will be closed upward in the c.e. degrees.

If this decomposition process continues *ad infinitum*, the theorem does not apply. We characterize these types of linear orderings (R_1, R_2, R_3, and R_4) in the following corollary.

Corollary 2.4. *Let M be a computable linear ordering with infinitely many successor pairs. If $M \not\cong R_i$ for $1 \leq i \leq 4$, where R_i is given below, then $DgSp_M(Succ_M)$ is closed upward in the c.e. degrees.*

Here, F_1, F_2 are arbitrary (possibly empty) linear orderings with finitely many successor pairs, $n_i, n_i' \in \omega$ are finite blocks of the appropriate size, and R may be any linear ordering:

$R_1 = F_1 + \omega + R + \omega^* + F_2,$
$R_2 = n_0 + \eta + n_1 + \eta + \cdots + R + \omega^* + F_2,$
$R_3 = F_1 + \omega + R + \cdots + \eta + n_1' + \eta + n_0',$
$R_4 = n_0 + \eta + n_1 + \eta + \cdots + R + \cdots + \eta + n_1' + \eta + n_0'.$

Proof. Let L be a computable linear ordering for which (U) does not hold. We decompose L as described above to obtain

$$L = R + \ldots + L_n + \ldots + L_2 + L_1,$$

where each L_i is of type 1 or $1 + \eta$.

Case 1. If for some k, and all $i > k$, L_i is of type 1, then $R + \ldots + L_n + \ldots + L_{k+1}$ is of type $R + \omega^*$. The remaining part of the decomposition, $F = L_k + L_{k-1} + \ldots + L_2 + L_1$ is a finite sum of orderings of type 1 or $1 + \eta$, and so F can have only finitely many successor pairs. In this case, we have

$$L = R + \omega^* + F,$$

where F is a linear ordering having finitely many successor pairs.

Case 2. If Case 1 does not hold, then for each k, there is $j > k$ so that L_j is of type $1 + \eta$. Hence, at most finitely many blocks of type 1 may appear consecutively. If n such blocks appear, they may be represented as a single block, n. In this case, we have

$$L = R + \ldots + \eta + n_1 + \eta + n_0,$$

where the n_i's are appropriate finite blocks.

Upon recalling that for any linear ordering L, $DgSp_L(Succ_L) = DgSp_{L^*}(Succ_{L^*})$, we arrive at the four decompositions above. □

In Theorem 1.2, the computable linear ordering L is constructed so that $DgSp_L(Succ_L) \subseteq \mathcal{R}(\geq deg(C))$. Because of its form, this ordering satisfies the condition (U) in Theorem 2.1, and we have the following.

Theorem 2.5. *For any c.e. degree* $\mathbf{a}$*, there is a linear ordering* L *so that the degree spectrum of* $Succ_L$ *is exactly the upper cone of c.e degrees determined by* $\mathbf{a}$*, that is,* $DgSp_L(Succ_L) = \mathcal{R}(\geq \mathbf{a})$.

Proof. If $\mathbf{a}$ is computable then the result follows from Example 1.3. Let $\mathbf{a}$ be a non-computable c.e. degree. Theorem 1.2 yields a linear ordering L with $deg(Succ_L) = \mathbf{a}$ that satisfies condition (U) of Theorem 2.1, and provides that $DgSp_L(Succ_L)$ is contained in the cone above $Succ_L$. Theorem 2.1 says that $DgSp_L(Succ_L)$ contains that cone. □

It will be interesting to investigate whether there is a computable linear ordering for which the successor relation is intrinsically incomplete,[2] in particular, whether the degree spectrum of the successor relation can consist of a single degree different from $\mathbf{0}$ and $\mathbf{0}'$ (see [3]). On the other hand, a similar question for the degree spectrum of the atom relation of computable Boolean algebras with infinitely many atoms was resolved by Downey and Remmel. Remmel [10] established that such a spectrum is closed upward in the c.e. degrees, and Downey [2] showed that such a spectrum must contain an incomplete degree.

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